THE REPRESENTATION THEORY OF C^*-ALGEBRAS ASSOCIATED TO GROUPOIDS

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Abstract. Let E be a second-countable, locally compact, Hausdorff groupoid equipped with an action of $\mathbb{T}$ such that $G := E/\mathbb{T}$ is a principal groupoid with Haar system λ. The twisted groupoid C^*-algebra $C^*(E; G, \lambda)$ is a quotient of the C^*-algebra of E obtained by completing the space of $\mathbb{T}$-equivariant functions on E. We show that $C^*(E; G, \lambda)$ is postliminal if and only if the orbit space of G is T_0 and that $C^*(E; G, \lambda)$ is liminal if and only if the orbit space is T_1. We also show that $C^*(E; G, \lambda)$ has bounded trace if and only if G is integrable and that $C^*(E; G, \lambda)$ is a Fell algebra if and only if G is Cartan.

Let $\mathcal{G}$ be a second-countable, locally compact, Hausdorff groupoid with Haar system λ and continuously varying, abelian isotropy groups. Let $\mathcal{A}$ be the isotropy groupoid and $\mathcal{R} := \mathcal{G}/\mathcal{A}$. Using the results about twisted groupoid C^*-algebras, we show that the C^*-algebra $C^*(\mathcal{G}, \lambda)$ has bounded trace if and only if $\mathcal{R}$ is integrable and that $C^*(\mathcal{G}, \lambda)$ is a Fell algebra if and only if $\mathcal{R}$ is Cartan. We illustrate our theorems with examples of groupoids associated to directed graphs.

1. Introduction

Let H be a locally compact, Hausdorff group acting continuously on a locally compact, Hausdorff space X. When the orbit space X/H is reasonable, for example if X/H is T_0, then every irreducible representation of the transformation-group C^*-algebra $C_0(X) \rtimes H$ is induced from an irreducible representation of an isotropy subgroup $S_x = \{h \in H : h \cdot x = x\}$. In particular, if the action of H on X is free then the spectrum of $C_0(X) \rtimes H$ is homeomorphic to the orbit space by [12], or if H is abelian then the spectrum of $C_0(X) \rtimes H$ is homeomorphic to a quotient of $(X/H) \times \hat{H}$ by [28]. Many of the postliminal (Type I) properties of the transformation-group C^*-algebra can be deduced from the dynamics of the transformation group (H, X). For example, $C_0(X) \rtimes H$ is postliminal if and only if the orbit space is T_0 and all the isotropy subgroups are postliminal [11]. There are many more results of this nature in the literature: [12, 29, 8] investigate when $C_0(X) \rtimes H$ has continuous trace, [15, 16, 1] when $C_0(X) \rtimes H$ is a Fell algebra or has bounded trace, and [28] when $C_0(X) \rtimes H$ is liminal. Usually the results are first proved for free actions and then generalized to non-free actions; but even when the isotropy groups are abelian the level of technical difficulty is much greater, and to get general results assumptions on the isotropy subgroups (for example, amenability or that they vary continuously) often seem unavoidable. The theorems above have served as a template for establishing similar

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theorems for the C^*-algebras of directed graphs [9, 13] and the C^*-algebras of groupoids [19, 20, 21, 4, 5, 6].

Let $\mathcal{G}$ be a locally compact, Hausdorff groupoid with abelian isotropy subgroups, and let $\mathcal{A}$ be the isotropy groupoid. The main theorem of [21] says that $C^*(\mathcal{G})$ has continuous trace if and only if the isotropy groups vary continuously and $\mathcal{G}/\mathcal{A}$ is a proper groupoid. The proof strategy, quickly, is to show that all the irreducible representations of $C^*(\mathcal{G})$ are induced, and to use the dual isotropy groupoid $\hat{\mathcal{A}}$ to construct a $\mathbb{T}$-groupoid whose associated twisted groupoid C^*-algebra is isomorphic to $C^*(\mathcal{G})$. Then the characterization of when twisted groupoid C^*-algebras have continuous trace from [20] completes the proof.

In this paper we generalize first the results from [20] to characterize when a twisted groupoid C^*-algebra has bounded trace or is a Fell algebra (Theorems 4.3 and 5.2), and second, the results from [21] to characterize when a groupoid with continuously varying, abelian isotropy groups has bounded trace or is a Fell algebra (Theorems 6.4 and 6.5). To do this we had to deal with non-Hausdorff spectra, which led to a sharpening of [20, Proposition 3.3] and [21, Proposition 4.5] (see Theorem 3.4 and Proposition 6.3). Theorem 3.4 says that when the orbit space of the groupoid is T_0, then the spectrum of the twisted groupoid C^*-algebra is homeomorphic to the orbit space and Proposition 6.3 establishes the isomorphism of $C^*(\mathcal{G})$ with the twisted groupoid C^*-algebra of [21] mentioned above under weaker hypothesis. Finally we illustrate our theorems with examples of groupoids associated to directed graphs.

2. Preliminaries

Let G be a locally compact, Hausdorff groupoid. We denote the unit space of G by $G^{(0)}$, the range and source maps $r, s : G \to G^{(0)}$ are $r(\gamma) = \gamma\gamma^{-1}$ and $s(\gamma) = \gamma^{-1}\gamma$, respectively, and the set of composable pairs by $G^{(2)}$. Recall that G is *principal* if the map $\gamma \mapsto (r(\gamma), s(\gamma))$ is injective.

Let $N \subseteq G^{(0)}$. The *saturation* of N is $r(s^{-1}(N)) = s(r^{-1}(N))$, and if $N = r(s^{-1}(N))$ then we say that N is *saturated.* We define the *restriction of* G *to* N to be $G|_N := \{\gamma \in G : s(\gamma) \in N \text{ and } r(\gamma) \in N\}$. The latter is not to be confused with $G_N := \{\gamma \in G : s(\gamma) \in N\}$. If $u \in G^{(0)}$, we call the saturation of $\{u\}$ the *orbit* of $u \in G^{(0)}$ and denote it by $[u]$; we also write G_u instead of $G_{\{u\}}$.

2.1. **$\mathbb{T}$-groupoids.** A $\mathbb{T}$-*groupoid* E is a topological groupoid E with a continuous free action of the circle group $\mathbb{T}$ on E such that

(1) if $(\gamma_1, \gamma_2) \in E^{(2)}$ and $s, t \in \mathbb{T}$ then

$$(s\gamma_1, t\gamma_2) \in E^{(2)} \quad \text{and} \quad (s\gamma_1)(t\gamma_2) = (ts)(\gamma_1\gamma_2);$$

(2) $G := E/\mathbb{T}$ is a principal groupoid.

In what follows, we will always assume that E is second-countable, locally compact and Hausdorff. Note that the composibility condition (1) implies that $s(\gamma) = s(t \cdot \gamma)$ and $r(\gamma) = r(t \cdot \gamma)$ for all $\gamma \in E$ and $t \in \mathbb{T}$; in particular, $E^{(0)} = G^{(0)}$. That G is principal implies that there is an exact sequence

$$E^{(0)} \longrightarrow E^{(0)} \times \mathbb{T} \xrightarrow{i} E \xrightarrow{j} G \longrightarrow E^{(0)} \tag{2.1}$$

where i is the homeomorphism $i(u,t) = t \cdot u$ onto a closed subgroupoid and j is the quotient map. Conversely, starting with a sequence (2.1), there is a free action of $\mathbb{T}$ on E defined by $t \cdot \gamma = i(r(\gamma), t)\gamma$, and the quotient $E/\mathbb{T}$ can be identified with G.

Remark 2.1. Since $\mathbb{T}$ is compact, $G = E/\mathbb{T}$ is Hausdorff, and since $\mathbb{T}$ is a compact Lie group, E is a locally trivial bundle over G by [23, Proposition 4.65 and Hooptedoodle 4.68]. That the sequence (2.1) is exact is equivalent to: every γ in the isotropy groupoid

$$\mathcal{A} := \{\gamma \in E : s(\gamma) = r(\gamma)\}$$

can be written as $t \cdot u$ for some $t \in \mathbb{T}$ and $u \in E^{(0)}$. Thus our $\mathbb{T}$-groupoid is what is called a "proper $\mathbb{T}$-groupoid" in [17, Definition 2.2]. But since we do not assume that $G = E/\mathbb{T}$ is étale, E is not a "twist" in the sense of [17, Definition 2.4]; $\mathbb{T}$-groupoids are more general. In particular, our assumption that E is $\mathbb{T}$-groupoid such that $G = E/\mathbb{T}$ is a principal groupoid puts us in the situation of [20].

Construction of the twisted groupoid C^*-algebra. We briefly outline the construction of the twisted groupoid C^*-algebra from [20]. Let E be a $\mathbb{T}$-groupoid over a principal groupoid G equipped with a left Haar system $\{\lambda^u : u \in G^{(0)}\}$. Then there is a left Haar system $\{\sigma^u : u \in E^{(0)} = G^{(0)}\}$ on E characterized by

$$\int_E f(\alpha)\, d\sigma^u(\alpha) = \int_G \int_{\mathbb{T}} f(t \cdot \alpha)\, dt\, d\lambda^u(j(\alpha)) \quad (f \in C_c(E)). \tag{2.2}$$

A left Haar system $\{\lambda^u : u \in G^{(0)}\}$ gives a right Haar system $\{\lambda_u : u \in G^{(0)}\}$ via $\lambda_u(E) := \lambda^u(E^{-1})$, and we will move freely between the left and right systems when convenient.

The usual groupoid C^*-algebra $C^*(E,\sigma)$ of E is the C^*-algebra which is universal for continuous nondegenerate $*$-representations $L : C_c(E) \to B(\mathcal{H}_L)$, where $C_c(E)$ has the inductive limit topology, $B(\mathcal{H}_L)$ the weak operator topology, and $C_c(E)$ is a $*$-algebra via

$$f * g(\gamma) = \int_E f(\gamma\alpha) g(\alpha^{-1})\, d\sigma^{s(\gamma)}(\alpha) \quad \text{and} \quad f^*(\gamma) = \overline{f(\gamma^{-1})}$$

for $f, g \in C_c(E)$.

The *twisted groupoid C^*-algebra* $C^*(E; G, \lambda)$ is a quotient of $C^*(E,\sigma)$ obtained as follows. Let $C_c(E;G)$ be the collection of $f \in C_c(E)$ such that $f(t \cdot \gamma) = t f(\gamma)$. Note that for $f, g \in C_c(E;G)$ and $\gamma \in E$, the function $\alpha \mapsto f(\gamma\alpha)g(\alpha^{-1})$ depends only on the class $j(\alpha)$ of α. So we can equip $C_c(E;G)$ with a *-algebra structure via

$$f * g(\gamma) = \int_G f(\gamma\alpha) g(\alpha^{-1})\, d\lambda^{s(\gamma)}(j(\alpha)) \quad \left(= \int_E f(\gamma\alpha) g(\alpha^{-1})\, d\sigma^{s(\gamma)}(\alpha) \right)$$

and $f^*(\gamma) = \overline{f(\gamma^{-1})}$ for $f, g \in C_c(E;G)$; using (2.2) it is straightforward to check that the formulae for $f * g$ in $C_c(E)$ and $C_c(E;G)$ coincide. Let $\operatorname{Rep}(E;G)$ be the collection of non-degenerate $*$-representations $L : C_c(E;G) \to B(\mathcal{H}_L)$ which are continuous when $C_c(E;G)$ has the inductive limit topology and $B(\mathcal{H}_L)$ has the weak operator topology. It follows from [26, Proposition 3.5 and Thèoréme 4.1] that for $f \in C_c(E;G)$

$$\|f\| = \sup\{\|L(f)\| : L \in \operatorname{Rep}(E;G)\}$$

is finite and defines a pre-C^*-norm on $C_c(E;G)$. The completion of $C_c(E;G)$ in this norm is the twisted groupoid C^*-algebra $C^*(E;G,\lambda)$. That $C^*(E;G,\lambda)$ is a quotient of $C^*(E,\sigma)$

follows because $\mathrm{Rep}(E;G)$ is a subset of the representations considered when constructing $C^*(E,\sigma)$. By Lemma 3.3 of [26], the surjective homomorphism $\Upsilon : C_c(E) \to C_c(E;G)$ defined by

$$\Upsilon(f)(\gamma) = \int_{\mathbb{T}} f(t\cdot\gamma)\bar{t}\,dt \tag{2.3}$$

is continuous in the inductive limit topology, and hence extends to a homomorphism $\Upsilon : C^*(E,\sigma) \to C^*(E;G,\lambda)$ called the quotient map. The reasons for calling $C^*(E;G,\lambda)$ the "twisted groupoid C^*-algebra" are outlined in [20, §2].

2.2. **Postliminal properties of C^*-algebras.** Let A be a C^*-algebra and $\hat{A}$ its spectrum. If π is an irreducible representation of A then we write $\mathcal{H}_\pi$ for the Hilbert space on which $\pi(A)$ acts. If $\pi(A) \supseteq K(\mathcal{H}_\pi)$ for every irreducible representation π of A, then A is *postliminal*; if $\pi(A) = K(\mathcal{H}_\pi)$ for every irreducible representation π of A, then A is *liminal*. In the literature postliminal and liminal C^*-algebras are also called GCR and CCR C^*-algebras, respectively. A positive element $b \in A$ is called a *bounded-trace element* if the map $[\pi] \mapsto \mathrm{tr}(\pi(b))$ is bounded on $\hat{A}$. Then A has *bounded trace* if the ideal consisting of the linear span of bounded-trace elements is dense in A. An irreducible representation π of A satisfies *Fell's condition* if there is a positive $a \in A$ and a neighbourhood U of $[\pi]$ in $\hat{A}$ such that $\sigma(a)$ is a rank-one projection whenever $[\sigma] \in U$. If every irreducible representation of A satisfies Fell's condition then A is a *Fell algebra*. A Fell algebra A has Hausdorff spectrum if and only if A has continuous trace. Each of the properties above are listed in order of reverse containment.

3. The spectrum of a twisted groupoid C^*-algebra

We start by investigating ideals in $C^*(E;G,\lambda)$ associated to open saturated subsets of the unit space of G.

Lemma 3.1. *Suppose that E is a second-countable, locally compact, Hausdorff, $\mathbb{T}$-groupoid such that $G := E/\mathbb{T}$ is a principal groupoid with Haar system λ. Let σ be the Haar system on E defined at* (2.2) *and U an open saturated subset of $G^{(0)}$ with $F := G^{(0)} \setminus U$. Then the short exact sequence*

$$0 \to C^*(E|_U,\sigma) \xrightarrow{i} C^*(E,\sigma) \xrightarrow{p} C^*(E|_F,\sigma) \to 0 \tag{3.1}$$

of [21, Lemma 2.10] *induces a short exact sequence*

$$0 \to C^*(E|_U;G|_U,\lambda) \xrightarrow{k} C^*(E;G,\lambda) \xrightarrow{r} C^*(E|_F;G|_F,\lambda) \to 0 \tag{3.2}$$

such that k is isometric and $\Upsilon\circ i = k\circ\Upsilon$ and $\Upsilon\circ p = r\circ\Upsilon$. On continuous functions the maps k and p are extension by 0 *and restriction, respectively.*

Proof. Note that we write just Υ for both the homomorphisms $\Upsilon : C^*(E,\sigma) \to C^*(E;G,\lambda)$ and $\Upsilon : C^*(E|_U,\sigma) \to C^*(E|_U;G|_U,\lambda)$. Since $\Upsilon\circ i = \Upsilon\circ i\circ\Upsilon$ on $C_c(E|_U)$ we have $\ker\Upsilon \subseteq \ker(\Upsilon\circ i)$, and hence there exists a unique homomorphism $k : C^*(E|_U;G|_U,\lambda) \to C^*(E;G,\lambda)$ such that $\Upsilon\circ i = k\circ\Upsilon$. Similarly, $\Upsilon\circ p = \Upsilon\circ p\circ\Upsilon$ on $C_c(E)$, so $\ker\Upsilon \subseteq \ker(\Upsilon\circ p)$, and hence there exists a unique homomorphism $r : C^*(E;G,\lambda) \to C^*(E|_F;G|_F,\lambda)$ such that $r\circ\Upsilon = \Upsilon\circ p$. Note that r is surjective because p and Υ are.

To see that k is isometric, fix a representation π of $C^*(E|_U; G|_U, \lambda)$. It suffices to see that π determines a representation $\hat{\pi}$ of $C_c(E; G)$ such that $\|\pi(f)\| = \|\hat{\pi}(k(f))\|$ for $f \in C_c(E|_U; G|_U)$; this will give $\|k(f)\| \geq \|f\|$ and hence $\|k(f)\| = \|f\|$.

By [21, Lemma 2.10], i is an isometric isomorphism of $C^*(E|_U)$ onto an ideal I of $C^*(E)$. Let $\tilde{\pi} : C^*(E) \to B(\mathcal{H}_\pi)$ be the canonical extension of $\pi \circ \Upsilon \circ i^{-1} : I \to B(\mathcal{H}_\pi)$. Note that, for $g \in C_c(E)$ and $h \in C_c(E|_U) \subseteq C_c(E)$ we have

$$\begin{aligned}\tilde{\pi}(\Upsilon(g))\pi \circ \Upsilon \circ i^{-1}(h) &= \pi \circ \Upsilon \circ i^{-1}(\Upsilon(g)h) = \pi \circ k^{-1} \circ \Upsilon(\Upsilon(g)h) \\ &= \pi \circ k^{-1} \circ \Upsilon(gh) = \pi \circ \Upsilon \circ i^{-1}(gh) \\ &= \tilde{\pi}(g)\pi \circ \Upsilon \circ i^{-1}(h).\end{aligned}$$

Thus $\tilde{\pi}(\Upsilon(g)) = \tilde{\pi}(g)$ and hence $\tilde{\pi}$ factors through $C^*(E; G, \lambda)$ and gives a representation $\hat{\pi} : C^*(E; G, \lambda) \to B(\mathcal{H}_\pi)$ such that $\tilde{\pi} = \hat{\pi} \circ \Upsilon$. Finally, if $f \in C_c(E|_U; G|_U)$ then for all $h \in C_c(E|_U)$ we have

$$\hat{\pi}(k(f))\pi \circ \Upsilon \circ i^{-1}(h) = \pi \circ \Upsilon \circ i^{-1}(k(f)h) = \pi \circ \Upsilon(fh) = \pi(f)\pi \circ \Upsilon \circ i^{-1}(h),$$

and hence $\hat{\pi}(k(f)) = \pi(f)$. Thus k is isometric.

Since k is isometric, the image of k is the completion of $C_c(E|_U; G|_U)$ viewed as functions on E. Since U and F are disjoint $r(C_c(E|_U; G|_U)) = 0$ and hence $\operatorname{range} k \subseteq \ker r$. Conversely, if $h \in C_c(E) \cap \ker r$ then h has support in U and hence is in the range of i. Thus $\operatorname{range} k = \ker r$. □

Fix $u \in G^{(0)}$ and let $\mathcal{H}_u^0$ be the collection of bounded Borel functions f on E with compact support in $E_u = s^{-1}(\{u\})$ satisfying $f(t \cdot \gamma) = tf(\gamma)$ for all $t \in \mathbb{T}$ and $\gamma \in E$. For each $\xi, \eta \in \mathcal{H}_u^0$ define

$$(\xi \mid \eta)_u = \int_G \xi(\gamma)\overline{\eta(\gamma)}\, d\lambda_u(j(\gamma)) \qquad \left(= \int_E \xi(\gamma)\overline{\eta(\gamma)}\, d\sigma_u(\gamma) \right)$$

to get an inner product on $\mathcal{H}_u^0$. Denote by $\mathcal{H}_u$ the Hilbert space completion of $\mathcal{H}_u^0$ with respect to this inner product; note that $\mathcal{H}_u$ is a closed subspace of $L^2(E_u, \sigma_u)$. Moreover, the functions obtained by restricting elements of $C_c(E; G)$ to E_u form a dense subset of $\mathcal{H}_u$ (see [20, Page 133]).

Let $f, \xi \in C_c(E; G)$. By [20, §3], the formula

$$L^u(f)\xi(\gamma) = f * \xi(\gamma) = \int_G f(\gamma\alpha)\xi(\alpha^{-1})\, d\lambda^u(j(\alpha)) \tag{3.3}$$

defines an appropriately continuous representation $L^u(E; G) = L^u$ of $C_c(E; G)$ on a dense subspace of $\mathcal{H}_u$, whence L^u extends to a representation L^u of $C^*(E; G, \lambda)$ on $\mathcal{H}_u$. By [20, Lemma 3.2], L^u is irreducible, and if $[u] = [v]$ then L^u and L^v are unitarily equivalent.

In Proposition 3.3 of [20], Muhly and Williams prove that if $C^*(E; G, \lambda)$ has Hausdorff spectrum, then $L : u \mapsto [L^u]$ induces a homeomorphism Ψ from the orbit space $G^{(0)}/G$ onto the spectrum of $C^*(E; G, \lambda)$; it seems from the application of [20, Proposition 3.3] in the proof of [21, Proposition 4.5] that its authors knew that the proof goes through using only that $C^*(E; G, \lambda)$ has T_1 spectrum (see [21, middle of p. 3638] and the applications of [21, Proposition 4.5] in the proof of [21, Theorem 1.1]).

The original proof of [20, Proposition 3.3] refers the reader to the proof of [19, Proposition 25] to see that Ψ induces a continuous injection; since the notations of [20] and [19]

don't match up, we had to carefully go through the details to verify that the Hausdorff condition wasn't needed, and we record these details here. The proof that Ψ is open onto its range given in [20, Proposition 3.3] used that $C^*(E;G,\lambda)$ has T_1 spectrum; our argument below does not require this hypothesis. We strengthen Proposition 3.2 further in Theorem 3.4 below.

Proposition 3.2 (Muhly-Williams). *Let E be a second-countable, locally compact, Hausdorff, $\mathbb{T}$-groupoid such that $G := E/\mathbb{T}$ is a principal groupoid with Haar system λ. For $u \in G^{(0)}$ let L^u be the irreducible representation defined at* (3.3).

(1) *Then the map $u \mapsto [L^u]$ induces a continuous injection $\Psi : G^{(0)}/G \to C^*(E;G,\lambda)^\wedge$ which is open onto its range.*
(2) *If $G^{(0)}/G$ is T_1 then Ψ is a homeomorphism of $G^{(0)}/G$ onto $C^*(E;G,\lambda)^\wedge$.*

Proof. (1) We start by showing that Ψ is continuous. Fix $\xi, \eta \in C_c(E;G)$ and $a \in C^*(E;G,\lambda)$. We claim that the map $x \mapsto (L^x(a)\xi \,|\eta)_x$ is continuous. To see why this is so, first consider $f \in C_c(E;G)$ and note that

$$(L^x(f)\xi \,|\eta)_x = \int_G (f * \xi)(\gamma)\overline{\eta(\gamma)}\, d\lambda_x(j(\gamma)),$$

where the convolution $f * \xi$ is taking place in $C_c(E;G)$. Since $(f*\xi)(\gamma)\overline{\eta(\gamma)}$ has compact support, the continuity of $x \mapsto (L^x(f)\xi \,|\eta)_x$ follows from the continuity of the Haar system. It now follows from an $\epsilon/3$ argument that the map $x \mapsto (L^x(a)\xi \mid \eta)_x$ is also continuous.

Now suppose that $u_n \to u$ in $G^{(0)}$; we will show by way of contradiction that $[L^{u_n}] \to [L^u]$. Suppose that $[L^{u_n}]$ does not converge to $[L^u]$. Then there exists a neighbourhood O of $[L^u]$ such that $[L^{u_n}] \notin O$ frequently. By passing to a subsequence and relabeling we may assume $[L^{u_n}] \notin O$ for all n. Let J be the ideal of $C^*(E;G,\lambda)$ such that $O = \{\rho \in C^*(E;G,\lambda)^\wedge \mid \rho(J) \neq 0\}$. So there exists $a \in J$ such that and $L^u(a) \neq 0$ and $L^{u_n}(a) = 0$ for all n. Now choose functions $\xi, \eta \in C_c(G)$ such that

$$(L^u(a)\xi \mid \eta)_u \neq 0;$$

but now

$$0 = (L^{u_n}(a)\xi \,|\, \eta)_{u_n} \to (L^u(a)\xi \,|\, \eta)_u \neq 0,$$

contradicting the continuity of $x \mapsto (L^x(a)\xi \mid \eta)_x$. So $[L^{u_n}] \to [L^u]$, and it follows that Ψ is continuous.

Next we show that Ψ is injective. Let $u, v \in G^{(0)}$ and suppose that L^u and L^v are unitarily equivalent. We will show that $[u] = [v]$. By [20, Lemma 3.1] there is a homomorphism $R : C_0(G^{(0)}) \to M(C^*(E;G,\lambda))$ defined by

$$(R(\phi)f)(\gamma) = \phi(r(\gamma))f(\gamma)$$

for $f \in C_c(E;G)$. In the proof of [20, Lemma 3.2] Muhly and Williams show that L^u is unitarily equivalent to a representation $T^u : C^*(E;G,\lambda) \to B(L^2([u], \mu_{[u]})$, and that $N_u := \overline{T^u} \circ R : C_0(G^{(0)}) \to B(L^2([u], \mu_{[u]})$ has formula $N_u(\phi)\eta(x) = \phi(x)\eta(x)$ for $\eta \in L^2([u], \mu_{[u]})$. Since L^u and L^v are unitarily equivalent so are T^u and T^v, and thus so are N_u and N_v. But now if $[u] \neq [v]$ then $[u] \cap [v] = \emptyset$ and [28, Lemma 4.15] implies that N^u and N^v are not unitarily equivalent, a contradiction. So $[u] = [v]$, and hence Ψ is injective.

To see that Ψ is open onto its range we show that Ψ^{-1} : range $\Psi \to G^{(0)}/G$ is continuous. We argue by contradiction: suppose that $[L^{u_n}] \to [L^u]$ in $C^*(E;G,\lambda)$ but $[u_n] \not\to [u]$ in $G^{(0)}/G$. Let U_0 be an open neighbourhood of $[u]$ in $G^{(0)}/G$; let $q : G^{(0)} \to G^{(0)}/G$ be the quotient map and set $U = q^{-1}(U_0)$. By passing to a subsequence and relabeling we may assume that $u_n \notin U$ for all n. By Lemma 3.1 $C^*(E|_U; G|_U)$ is isomorphic to an ideal I of $C^*(E;G)$. Set $F := G^{(0)} \setminus U$. Fix $f \in C_c(E;G)$ such that $f(\gamma) = 0$ for $\gamma \in E|_F$ (such f are dense in I) and fix $\xi \in \mathcal{H}_{u_n}$. Then

$$\begin{aligned}\|L^{u_n}(f)\xi\|_{u_n}^2 &= \int_G |L^{u_n}(f)\xi(\gamma)|^2 \, d\lambda_{u_n}(j(\gamma))\\ &= \int_G \Big(\int_G f(\gamma\alpha^{-1})\xi(\alpha)\, d\lambda_{u_n}(j(\alpha))\Big)^2 \, d\lambda_{u_n}(j(\gamma)).\end{aligned}$$

Now consider the inner integrand: there $r(\gamma\alpha^{-1}) = r(\gamma)$ and $s(\gamma) = u_n$, so $\gamma\alpha^{-1} \in E|_{[u_n]}$. But U is saturated with $u_n \notin U$, so $\gamma\alpha^{-1} \in E|_F$ and $f(\gamma\alpha^{-1}) = 0$. Thus $\|L^{u_n}(f)\xi\|_{u_n}^2 = 0$, and since ξ was fixed we have $L^{u_n}(f) = 0$. It now follows that $I \subseteq \ker L^{u_n}$ for all n. But now $[L^{u_n}] \notin \hat{I}$ for all n, contradicting that $\hat{I}$ is an open neighbourhood of $[L^u]$ and $[L^{u_n}] \to [L^u]$. Thus Ψ is open.

(2) In view of (1) it suffices to show that Ψ is surjective. See the proof of [20, Proposition 3.3] (the proof given there only uses that $G^{(0)}/G$ is T_1). □

Proposition 3.3. *Suppose that E is a second-countable, locally compact, Hausdorff, $\mathbb{T}$-groupoid such that $G := E/\mathbb{T}$ is a principal groupoid with Haar system λ. Then*

(1) *$C^*(E;G,\lambda)$ is liminal if and only if the orbit space $G^{(0)}/G$ is T_1; and*
(2) *$C^*(E;G,\lambda)$ is postliminal if and only if the orbit space $G^{(0)}/G$ is T_0.*

Proof. Let σ be the Haar system for E from (2.2).

(1) Suppose $C^*(E;G,\lambda)$ is liminal. Since $C^*(E;G,\lambda)$ is separable, its spectrum is T_1 by [7, 9.5.3]. By Proposition 3.2(1), $\Psi : G^{(0)}/G \to C^*(E;G,\lambda)^\wedge$, $[u] \mapsto [L^u]$ is a continuous injection. So for each $u \in G^{(0)}$, $\{[u]\} = \Psi^{-1}(\{[L^u]\})$ is closed in $G^{(0)}/G$. Thus $G^{(0)}/G$ is T_1.

Conversely, suppose $G^{(0)}/G$ is T_1. Since all the isotropy groups of E are amenable, $C^*(E,\sigma)$ is liminal by [5, Theorem 6.1]. Now $C^*(E;G,\lambda)$ is liminal because quotients of liminal algebras are liminal.

(2) Proceed as in (1) using [7, 9.5.2] and [5, Theorem 6.1]. □

We now improve Proposition 3.2 by using a composition series to reduce to the T_1 case:

Theorem 3.4. *Suppose that E is a second-countable, locally compact, Hausdorff, $\mathbb{T}$-groupoid such that $G := E/\mathbb{T}$ is a principal groupoid with Haar system λ. If $G^{(0)}/G$ is T_0 then Ψ is a homeomorphism of $G^{(0)}/G$ onto $C^*(E;G,\lambda)^\wedge$.*

Proof. We adapt the argument of [4, Proposition 5.1]. By Proposition 3.2(1) it suffices to show that Ψ is onto. Since G is σ-compact the equivalence relation $R = \{(r(\gamma), s(\gamma)) : \gamma \in G\}$ is an F_σ set, so $G^{(0)}/G$ is almost Hausdorff by [24, Theorem 2.1]. A Zorn's lemma argument (see the discussion on page 25 of [10]) gives an ordinal γ and a collection $\{V_\alpha : \alpha \le \gamma\}$ of open subsets of $G^{(0)}/G$ such that $V_0 = \emptyset$, $V_\gamma = G^{(0)}/G$, $\beta < \alpha$ implies $V_\beta \subseteq V_\alpha$, if α is a limit ordinal then $V_\alpha = \cup_{\beta<\alpha} V_\beta$, and if α is not a limit ordinal or 0 then $V_\alpha \setminus V_{\alpha-1}$ is an open Hausdorff subset of $(G^{(0)}/G) \setminus V_{\alpha-1}$. Let $q : G^{(0)} \to G^{(0)}/G$

be the quotient map and set $U_\alpha := q^{-1}(V_\alpha)$. By Lemma 3.1, for each $\alpha \le \gamma$, there is an isomorphism k_α of $C^*(E|_{U_\alpha}; G|_{U_\alpha}, \lambda)$ onto an ideal I_α of $C^*(E; G, \lambda)$.

Now fix an irreducible representation π of $C^*(E; G, \lambda)$. Let α be the smallest element of the set $\{\lambda \le \gamma : \pi(I_\lambda) \ne 0\}$. Note that α is not a limit ordinal (if α were a limit ordinal then $U_\alpha = \cup_{\beta<\alpha} U_\beta$ and I_α is the ideal generated by the I_β with $\beta < \alpha$. But then $\pi(I_\alpha) \ne 0$ implies $\pi(I_\beta) \ne 0$ for some $\beta < \alpha$, contradicting the minimality of α.) So $\pi(I_\alpha) \ne 0$ and $\pi(I_{\alpha-1}) = 0$. It follows that π is the canonical extension of the representation $\pi|_{I_\alpha}$ and that $\pi|_{I_\alpha}$ factors through a representation of $I_\alpha/I_{\alpha-1}$. By Lemma 3.1, $I_\alpha/I_{\alpha-1}$ is isomorphic to $C^*(E|_{U_\alpha \setminus U_{\alpha-1}}; G|_{U_\alpha \setminus U_{\alpha-1}}, \lambda)$. The orbit space $V_\alpha \setminus V_{\alpha-1}$ of $G|_{U_\alpha \setminus U_{\alpha-1}}$ is Hausdorff, hence T_1. Now apply Proposition 3.2(2) to get that $\pi|_{I_\alpha}$ is unitarily equivalent $L^u(E|_{U_\alpha}; G|_{U_\alpha}) \circ k_\alpha^{-1}$ for some $u \in U_\alpha \setminus U_{\alpha-1}$. Note that $L^u(E|_{U_\alpha}; G|_{U_\alpha}) \circ k_\alpha^{-1} = L^u(E; G)|_{I_\alpha}$. Since $\pi|_{I_\alpha}$ is unitarily equivalent to $L^u(E|_{U_\alpha}; G|_{U_\alpha}) \circ k_\alpha^{-1}$ it follows that their canonical extensions are unitarily equivalent. Thus π is unitarily equivalent to $L^u(E; G)$. So Ψ is onto. □

4. The twisted groupoid C^*-algebras with bounded trace

We recall [6, Definition 3.1]: a locally compact, Hausdorff groupoid G is *integrable* if for every compact subset N of $G^{(0)}$,

$$\sup_{x\in N}\{\lambda^x(s^{-1}(N))\} < \infty. \tag{4.1}$$

Equivalently, by [6, Lemma 3.5], G is integrable if and only if for each $z \in G^{(0)}$, there exists an open neighbourhood U of z in $G^{(0)}$ such that

$$\sup_{x\in U}\{\lambda^x(s^{-1}(U))\} < \infty. \tag{4.2}$$

So if a groupoid fails to be integrable, then there exists a $z \in G^{(0)}$ so that

$$\sup_{x\in U}\{\lambda^x(s^{-1}(U))\} = \infty, \tag{4.3}$$

for all open neighbourhoods U of z, and we say G *fails to be integrable at z.*

In Proposition 4.2 we will prove that if G is integrable, then $C^*(E; G, \lambda)$ has bounded trace. To do so, we need to know that all irreducible representations of $C^*(E; G, \lambda)$ are equivalent to L^u for some $u \in G^{(0)}$. We proved in [6, Lemma 3.9] that if G is integrable, then all the orbits are closed; unfortunately, there is a gap in the proof (the proof assumes implicitly that orbits are locally closed at Equation 3.2 in [6]). Lemma 4.1 establishes that if G is a principal integrable groupoid, then the orbits are indeed locally closed. The proof of [6, Lemma 3.9] then goes through as written. The proof of Lemma 4.1 is based on the proof of [1, Lemma 2.1] which establishes similar results in the transformation-group setting.

Lemma 4.1. *Let G be a second-countable, locally compact, Hausdorff, principal groupoid and let $z \in G^{(0)}$.*

1. *If the orbit $[z]$ is not locally closed then for every open neighbourhood V of z in $G^{(0)}$, $\lambda_z(r^{-1}(V)) = \infty$.*
2. *If G is integrable then the orbits are locally closed.*

Proof. (1) Let W be any open neighbourhood of z in $G^{(0)}$. We claim that for every compact neighbourhood L of z in G there exists $\gamma_L \in G \setminus L$ such that $s(\gamma_L) = z$ and

$r(\gamma_L) \in W$. Suppose there exists an L for which no such γ exists. Since $[z]$ is not locally closed, $(\overline{[z]} \setminus [z]) \cap W \neq \emptyset$. Let $y \in (\overline{[z]} \setminus [z]) \cap W$. Then there exists $\{\gamma_i\} \subseteq G$ such that $s(\gamma_i) = z$, $r(\gamma_i) \to y$. Then $r(\gamma_i) \in W$ eventually. So by assumption, $\gamma_i \in L$ eventually. By passing to a subsequence we may assume that $\gamma_i \to \gamma \in L$. But now $s(\gamma) = z$ and $r(\gamma) = y$, and hence $y \in [z]$, a contradiction. This proves the claim.

Let V be an open neighbourhood of z in $G^{(0)}$ and let $M \in \mathbb{P}$. There exists an open neighbourhood U of z in G^0 and a compact symmetric neighbourhood K of z in G such that $r(KU) \subseteq V$. Let $c > 0$ such that $\lambda_u(K) \geq c$ for $u \in U$ by [**6**, Lemma 3.10(1)]. Choose $k \in \mathbb{P}$ such that $kc > M$ and $\gamma^{(1)}, \dots, \gamma^{(k)}$ as follows. Set $\gamma^{(1)} = z$. By the claim there exists $\gamma^{(2)} \in G \setminus (K^2\gamma^{(1)})$ such that $s(\gamma^{(2)}) = z$ and $r(\gamma^{(2)}) \in U$. Next, note that $K^2\gamma^{(1)} \cup K^2\gamma^{(2)}$ is compact, so by the claim there exists $\gamma^{(3)} \in G \setminus (K^2\gamma^{(1)} \cup K^2\gamma^{(2)})$ with $s(\gamma^{(3)}) = z$ and $r(\gamma^{(3)}) \in U$. Continue.

Now $r(\gamma^{(i)}) \in U$ for $1 \leq i \leq k$ and $\gamma^{(j)}(\gamma^{(i)})^{-1} \in G \setminus K^2$ when $i \neq j$. Thus $r(K\gamma^{(i)}) \subseteq r(KU) \subseteq V$ and $K\gamma^{(i)} \cap K\gamma^{(j)} = \emptyset$ when $i \neq j$. We have

$$\lambda_z(r^{-1}(V)) \geq \lambda_z\Big(\bigcup_{i=1}^{k} K\gamma^{(i)}\Big) = \sum_{i=1}^{k} \lambda_z(K\gamma^{(i)}) = \sum_{i=1}^{k} \lambda_{r(\gamma^{(i)})}(K) \geq kc > M.$$

Since M was arbitrary, $\lambda_z(r^{-1}(V)) = \infty$.

(2) Suppose there exists $z \in G^{(0)}$ such that $[z]$ is not locally closed. Let V be any open, relatively compact neighbourhood of$z$ in G. By (1), $\lambda_z(r^{-1}(V)) = \infty$ thus $\sup\{\lambda^x(s^{-1}(\overline{V})) : x \in \overline{V}\} \geq \sup\{\lambda_x(r^{-1}(V)) : x \in V\} = \infty$ so G fails to integrable at z. □

Proposition 4.2. *Suppose that E is a second-countable, locally compact, Hausdorff, $\mathbb{T}$-groupoid such that $G := E/\mathbb{T}$ is a principal groupoid with Haar system λ. If G is integrable then the twisted groupoid C^*-algebra $C^*(E; G, \lambda)$ has bounded trace.*

Proof. Since G is an integrable principal groupoid, the orbits of G are locally closed by Lemma 4.1. By Theorem 3.4, $[u] \mapsto [L^u]$ is a homeomorphism of $G^{(0)}/G$ onto the spectrum of $C^*(E; G, \lambda)$. Fix $u \in G^{(0)}$ and $f \in C_c(E; G)^+$. Then, by [20, Proposition 4.1], $L^u(f)$ is trace class with

$$\operatorname{tr}(L^u(f)) = \int_G f(r(\gamma))\, d\lambda_u(j(\gamma)).$$

Thus

$$\begin{aligned} \operatorname{tr}(L^u(f)) &\leq \|f\|_\infty \lambda_u\{j(\gamma) : r(\gamma) \in \operatorname{supp} f\} \\ &= \|f\|_\infty \lambda_u\{j(\gamma) : \gamma \in r^{-1}(\operatorname{supp} f)\} \\ &= \|f\|_\infty \lambda_u\{j(\gamma) : j(\gamma) \in j(r^{-1}(\operatorname{supp} f))\} \\ &= \|f\|_\infty \lambda_u\{j(\gamma) : j(\gamma) \in r^{-1}(j(\operatorname{supp} f))\} \\ &\leq \|f\|_\infty \sup_{u \in j(\operatorname{supp} f)} \{\lambda_u(r^{-1}(j(\operatorname{supp} f)))\} < \infty \end{aligned}$$

because $\operatorname{supp} f$ is compact and G is integrable. Thus $C_c(E; G)^+$ is contained in the ideal spanned by the bounded-trace elements, and hence $C^*(E; G, \lambda)$ has bounded trace. □

In Theorem 4.3 below we show that if $C^*(E; G, \lambda)$ has bounded trace then G is integrable. The proof is modeled on [20, Theorem 4.3], where Muhly and Williams prove that

if $C^*(E;G,\lambda)$ has continuous trace then G is a proper groupoid. Their proof strategy is the following. Suppose that G is not proper. Then G fails to be proper at some $z \in G^{(0)}$. This gives a sequence $\{x_n\} \subseteq G$ which is eventually disjoint from every compact subset of G and $r(x_n)$, $s(x_n) \to z$. (In the terminology of [6, Definition 3.6], $u_n := s(x_n)$ converges 2-times in $G^{(0)}/G$ to z.)

To show that $C^*(E;G,\lambda)$ does not have continuous trace, they construct a function F and show that F^*F has certain tracial properties. In particular, they partition the Hilbert space $\mathcal{H}_u$ of L^u into a direct sum $\mathcal{H}_{u,1} \oplus \mathcal{H}_{u,2}$ and write $P_{u,i}$ for the projection of $\mathcal{H}_u$ onto $\mathcal{H}_{u,i}$. First they show that $u \mapsto \operatorname{tr}(L^u(F^*F)P_{u,1})$ is continuous at z. Second, they have a really clever and technical argument to show that there is a constant $a > 0$ such that $\operatorname{tr}(L^{u_n}(F^*F)P_{u_n,2}) \geq \|L^{u_n}(F^*F)P_{u_n,2}\| \geq a$ eventually. Next they push F^*F into the Pedersen ideal of [22, Theorem 5.6.1] using a function $q \in C_c(0,\infty)$ such that $q(t) = t$ for $t \in [a, \|F^*F\|]$. This gives an element $d = q(F^*F)$ in the Pedersen ideal such that $\operatorname{tr}(L^{u_n}(d)P_{u_n,2}) \geq a > 0$ eventually. It follows that $[L^u] \mapsto \operatorname{tr}(L^u(d))$ is not continuous at $[L^z]$. Thus d is not a continuous-trace element. But the Pedersen ideal is the minimal dense ideal of a C^*-algebra, so the ideal spanned by the continuous-trace elements cannot be dense in $C^*(E;G,\lambda)$. Thus $C^*(E;G,\lambda)$ does not have continuous trace.

Theorem 4.3. *Suppose that E is a second-countable, locally compact, Hausdorff, $\mathbb{T}$-groupoid such that $G := E/\mathbb{T}$ is a principal groupoid with Haar system λ. The following are equivalent:*

1. *the twisted groupoid C^*-algebra $C^*(E;G,\lambda)$ has bounded trace;*
2. *G is integrable; and*
3. *$C^*(G)$ has bounded trace.*

Since G is principal, (2) and (3) are equivalent by [6, Theorem 4.4]. By Proposition 4.2, if G is integrable then $C^*(E;G,\lambda)$ has bounded trace, so it remains to show that (1) implies (2). We prove the contrapositive. So suppose that G is not integrable, say G fails to be integrable at $z \in G^{(0)}$. Then by [6, Proposition 3.11] there exists a sequence $\{u_n\}$ in $G^{(0)}$ so that $\{u_n\}$ converges to z, $u_n \neq z$ for all n, and u_n *converges k-times in $G^{(0)}/G$ to z*, for every $k \in \mathbb{P}$. That is, there exist k sequences $\{\gamma_n^{(1)}\}$, $\{\gamma_n^{(2)}\}, \ldots, \{\gamma_n^{(k)}\} \subseteq G$ such that

1. $r(\gamma_n^{(i)}) \to z$ and $s(\gamma_n^{(i)}) = u_n$ for $1 \leq i \leq k$;
2. if $1 \leq i < j \leq k$ then $\gamma_n^{(j)}(\gamma_n^{(i)})^{-1} \to \infty$ as $n \to \infty$, in the sense that $\{\gamma_n^{(j)}(\gamma_n^{(i)})^{-1}\}$ admits no convergent subsequence.

We will prove that $C^*(E;G,\lambda)$ does not have bounded trace.

Since C^*-algebras with bounded trace are liminal, we may assume that the orbits are closed in $G^{(0)}$ by Proposition 3.3. Let $M > 0$ be given. In order to show that $C^*(E;G,\lambda)$ does not have bounded trace, we will show that there exists an element d of the Pedersen ideal of $C^*(E;G,\lambda)$ such that $\operatorname{tr}(L^{u_n}(d)) > M$ eventually (see (3.3) for the definition of the irreducible representations L^{u_n}). Since M is arbitrary and the Pedersen ideal is the minimal dense ideal [22, Theorem 5.6.1], this shows that the ideal of bounded-trace elements cannot be dense.

We will use the same function F as Muhly and Williams and adapt their proof as follows.

1. Show that there exists a constant $a > 0$ such that $\|L^{u_n}(F^*F)P_{u_n,1}\| \geq a$.

(2) Fix $l \in \mathbb{N}$ such that $la > M$. Since $\{u_n\}$ converges k-times to z in $G^{(0)}/G$ for any k, there exist l sequences $\{u_n = \gamma_n^{(1)}\}, \{\gamma_n^{(2)}\}, ..., \{\gamma_n^{(l)}\} \subseteq G$ satisfying the items (1) and (2) listed above.
(3) Using that $\gamma_n^{(j)}(\gamma_n^{(i)})^{-1} \to \infty$, partition $\mathcal{H}_{u_n}$ into l summands $\mathcal{H}_{u_n,i}$.
(4) Write $P_{u_n,i}$ for the projection of $\mathcal{H}_{u_n}$ onto $\mathcal{H}_{u_n,i}$. Show that, for every $1 \leq i \leq l$, $\|L^{u_n}(F^*F)P_{u_n,i}\| \geq a$ eventually.
(5) Push F^*F into the Pedersen ideal.

Note that the order of events is subtle. We have to find the constant a before we can choose an appropriate l and then get l sequences in G which witness the l-times convergence of the sequence $\{u_n\}$. We retain, as much as possible, the notation of [20].

We start by explaining the function F (see (4.5) below). Fix a function $g \in C_c^+(G^{(0)})$ so that $0 \leq g \leq 1$ and g is identically one on a neighbourhood U of z. By [19, Lemma 2.7] there exist symmetric, open, conditionally compact neighbourhoods W_0 and W_1 in G such that $G^{(0)} \subseteq W_0 \subseteq \overline{W}_0 \subseteq W_1$ and $\overline{W_1}z \setminus W_0 z \subseteq r^{-1}(G^{(0)} \setminus \operatorname{supp}(g))$. Choose symmetric, relatively compact, open neighbourhoods V_0 and V_1 of z in G such that $\overline{V_0} \subseteq V_1$. Apply Lemma A.1 to obtain a compact neighbourhood A of z in $G^{(0)}$ such that

$$(\overline{W_1}^7 \overline{V_1} \setminus W_0 V_0) \cap G_A \subseteq r^{-1}(G^{(0)} \setminus \operatorname{supp} g).$$

Thus

$$(\overline{W_1}^7 \overline{V_1}\, \overline{W_1}^7 \setminus W_0 V_0 W_0) \cap G_A \subseteq r^{-1}(G^{(0)} \setminus \operatorname{supp} g). \tag{4.4}$$

For $\gamma \in E_A$, set

$$g^{(1)}(\gamma) = \begin{cases} g(r(\gamma)), & \text{if } j(\gamma) \in \overline{W_1}^7 \overline{V_1}\, \overline{W_1}^7 \\ 0, & \text{if } j(\gamma) \notin W_0 V_0 W_0, \end{cases}$$

where $j : E \to E/\mathbb{T}$ is the quotient map. This gives a well-defined function $g^{(1)}$ on E_A which, by Tietze's extension theorem extends to a well-defined function, also called $g^{(1)}$, in $C_c(E)$.

Next choose a self-adjoint $b \in C_c(G)$ such that $0 \leq b \leq 1$, b is identically one on $W_0 V_0 W_0^2 V_0 W_0$ and b vanishes off $\overline{W_1}^4 \overline{V_1}\, \overline{W_1}^4$. Also choose a compact neighbourhood C of z in $G^{(0)} = E^{(0)}$ such that $i(C \times \mathbb{T}) \supseteq G^{(0)} \cap \operatorname{supp} g^{(1)}$. Define $l : i(C \times \mathbb{T}) \to \mathbb{T}$ by $l(i(u,t)) = 1$ and extend l to a function $l \in C_c(E)$. By replacing l by $(l + l^*)/2$ we may assume l is self-adjoint. Set $h = \Upsilon(l)$ to obtain a self-adjoint $h \in C_c(E; G)$ such that $h(i(u,1)) = 1$ for all $u \in C$. Finally, define

$$F(\gamma) = g(r(\gamma))g(s(\gamma))b(j(\gamma))h(\gamma). \tag{4.5}$$

Note that the h ensures $F \in C_c(E; G)$, and that F is self-adjoint because h and b are.

For each n, define

$$\mathcal{H}_{u_n,1} = \mathcal{H}_{u_n} \cap L^2(E_{u_n} \cap j^{-1}(W_0 V_0 W_0), \sigma_{u_n})$$

and let $P_{u_n,1}$ be the projection onto $\mathcal{H}_{u_n,1}$. The calculation [20, pages 140–141] shows that $L^{u_n}(F)\mathcal{H}_{u_n,1} \subseteq \mathcal{H}_{u_n,1}$.

By [19, Lemma 2.9] there is a neighbourhood V_2 of z in G and a conditionally compact, symmetric neighbourhood Y of $G^{(0)}$ in G such that $V_2 \subseteq V_0$, and

$$\gamma \in V_2 \Longrightarrow r(Y\gamma) \subseteq U. \tag{4.6}$$

In particular, since $r(Y\gamma) = r(Yr(\gamma))$, we get $r(Y\gamma) \subseteq U$ whenever $r(\gamma) \in V_2$. Since $u_n \to z$ we may assume that $u_n \in V_2 \cap C$ for all n.

The following lemma closely resembles [20, Lemma 4.5] and our proof is similar; we replace the unbounded sequence $\{x_n\}$ appearing in [20, Lemma 4.5] with the sequence $\{u_n\}$ (corresponding to $\{r(x_n)\}$) which causes us to consider the projection onto $\mathcal{H}_{u_n,1}$ rather than the projection onto $\mathcal{H}_{u_n,1}^{\perp}$).

Lemma 4.4. *Let F be the function defined at* (4.5). *There exist an $a > 0$ and a neighbourhood $V_3 \subseteq V_2 \cap C$ of z in G such that*

$$\|L^{u_n}(F^*F)P_{u_n,1}\| \geq a$$

whenever $u_n \in V_3$.

Proof. Let Y be as above. Let $\mathcal{O}_1, \mathcal{O}_2, c, Y_0$ be as in [20]. Thus $\mathcal{O}_1$ and $\mathcal{O}_2$ are open neighbourhoods of $i(C \times 1)$ in E so that for every $\eta \in \mathcal{O}_1$, $\mathrm{Re}(h(\eta)) > \frac{1}{2}$ and for every $\eta \in \mathcal{O}_2$, $\mathrm{Re}(h(\eta)) > \frac{1}{4}$, and c is a regular cross section of j (see [20, Proof of Lemma 3.2] for definition of regular). The set Y_0 is a conditionally compact, symmetric neighbourhood Y_0 of $G^{(0)}$ in G such that $CY_0 \subseteq j(\mathcal{O}_1)$ and $Y \subseteq Y_0$.

Let $\mathbb{T}_0 = \{t_i\}_{i=1}^{\infty}$ be a countable dense subset of $\mathbb{T}$. If $x \in Yu_n$, then

$$j(c(u_n)c(x)^{-1}) = u_n x^{-1} \in CY \subseteq CY_0 \subseteq j(\mathcal{O}_1).$$

So there exists a $t \in \mathbb{T}_0$ so that $t \cdot c(u_n)c(x)^{-1} \in \mathcal{O}_1$. Define $\zeta_n : Yu_n \to \mathbb{T}_0$ by $\zeta_n(y) = t_j$ where $j = \min\{k : t_k \cdot c(u_n)c(y)^{-1} \in \mathcal{O}_1\}$. Thus apart from its domain ζ is the function defined in [20, Lemma 4.5]; that our ζ_n is Borel is proved as is done for the function in [20, Lemma 4.6].

By another argument very similar to that of [19, Lemma 2.9], there exists a conditionally compact, symmetric neighbourhood $\tilde{Y}$ of $E^{(0)}$ in E such that $\tilde{Y}j^{-1}(CY) \subseteq \mathcal{O}_2$ and $j(\tilde{Y}) = Y$. Since we are assuming that $u_n \in V_2 \cap C$ for all n, if $x \in Yu_n$ the claim gives

$$\tilde{Y}\zeta_n(x)u_n c(x)^{-1} \subseteq \tilde{Y}C\tilde{Y} \subseteq \tilde{Y}j^{-1}(CY) \subseteq \mathcal{O}_2 \tag{4.7}$$

for all n.

Muhly and Williams use a function $t : E \to \mathbb{T}$ defined as follows: for each $\gamma \in E$, consider the element $\gamma c(j(\gamma))^{-1}$. This is in the image of i and equals $i(u, s)$ for some $s \in \mathbb{T}$; then $t(\gamma) := s$. Let χ_n be the characteristic function of Yu_n and define $\xi_n : E \to \mathbb{C}$ by

$$\xi_n(\gamma) = t(\gamma)\zeta_n(j(\gamma))\chi_n(j(\gamma)).$$

Since all of the functions involved in defining ξ_n are Borel, so is ξ_n. Also, it is clear that ξ_n is bounded and has compact support contained in $\mathrm{supp}(\xi_n) \subseteq j^{-1}(\overline{Yu_n})$. Notice that $t(s \cdot \gamma) = st(\gamma)$ so that $\xi_n \in \mathcal{H}_{u_n}$; since also $Yu_n \subseteq W_0V_0W_0$ we have $\xi_n \in \mathcal{H}_{u_n,1}$. (This is where we have departed from the Muhly-Williams proof - their unbounded sequence $\{x_n\}$ used in place of our $\{u_n\}$ ensures their ξ_n has support in the orthogonal complement of $\mathcal{H}_{u_n,1}$.)

Now fix $\gamma \in \tilde{Y} u_n$ and compute:

$$
\begin{aligned}
L^{u_n}(F)(\xi_n)(\gamma) = F * \xi_n(\gamma) &= \int_G F(\gamma\alpha^{-1})\xi_n(\alpha)\ d\lambda_{u_n}(j(\alpha)) \\
&= \int_G g(r(\gamma\alpha^{-1}))g(s(\gamma\alpha^{-1}))b(j(\gamma\alpha^{-1}))h(\gamma\alpha^{-1})\xi_n(\alpha)\ d\lambda_{u_n}(j(\alpha)) \\
&= g(r(\gamma)) \int_G g(r(\alpha))b(j(\gamma\alpha^{-1}))h(\gamma\alpha^{-1})\xi_n(\alpha)\ d\lambda_{u_n}(j(\alpha)). \qquad (4.8)
\end{aligned}
$$

Note that the integrand is zero unless $j(\alpha) \in Y u_n$. Let $j(\alpha) \in Y u_n$; then $j(\gamma\alpha^{-1}) \in Y u_n u_n^{-1} Y \subseteq Y V_0 Y \subseteq W_0 V_0 W_0$, and hence $b(j(\gamma\alpha^{-1}) = 1$. Also $j(\operatorname{supp} \xi_n) \subseteq Y u_n$, so

$$
\begin{aligned}
(4.8) = g(r(\gamma)) &\int_{Y u_n} g(r(\alpha))h(\gamma\alpha^{-1})\xi_n(\alpha)\ d\lambda_{u_n}(j(\alpha)) \\
= g(r(\gamma)) &\int_{Y u_n} g(r(x))h(\gamma c(x)^{-1})\xi_n(c(x))\ d\lambda_{u_n}(x)
\end{aligned}
$$

by letting $x = j(\alpha)$ and noting that $r(\alpha) = r(x)$ and $c(x) = c(j(\alpha)) = \alpha$. Since $r(\tilde{Y} u_n) = r(Y u_n) \subseteq U$ by our choice of Y at (4.6) and since g is identically one on U, this is

$$
= \int_{Y u_n} h(\gamma c(x)^{-1})\xi_n(c(x))\ d\lambda_{u_n}(x). \qquad (4.9)
$$

By the definition of ξ_n and using that $t \circ c = 1$ we get $\xi_n(c(x)) = \zeta_n(x)$ for $x \in Y u_n$, and since h is $\mathbb{T}$-equivariant we get

$$
(4.9) = \int_{Y u_n} h(\gamma\zeta_n(x)c(x)^{-1})\ d\lambda_{u_n}(x).
$$

But for $x \in Y u_n$, $\gamma\zeta_n(x)c(x)^{-1} \in \mathcal{O}_2$ by (4.7), so $\operatorname{Re}(h(\gamma\zeta_n(x)c(x)^{-1})) > \frac{1}{4}$. So

$$
\operatorname{Re}\big(L^{u_n}(F)(\xi_n)(\gamma)\big) \geq \frac{1}{4}\lambda_{u_n}(Y u_n) = \frac{1}{4}\lambda_{u_n}(Y)
$$

and hence

$$
|L^{u_n}(F)(\xi_n)(\gamma)|^2 \geq \frac{1}{16}\lambda_{u_n}(Y)^2.
$$

Now

$$
\begin{aligned}
\|L^{u_n}(F)\xi_n\|^2 &= \int_G |L^{u_n}(F)(\xi_n)(\gamma)|^2\ d\lambda_{u_n}(j(\gamma)) \\
&\geq \int_{\{j(\gamma):\gamma\in\tilde{Y}u_n\}} |L^{u_n}(F)(\xi_n)(\gamma)|^2 d\lambda_{u_n}(j(\gamma)) \\
&\geq \int_{Y u_n} \frac{1}{16}\lambda_{u_n}(Y)^2\ d\lambda_{u_n}(j(\gamma)) = \frac{1}{16}\lambda_{u_n}(Y)^3.
\end{aligned}
$$

By [6, Lemma 3.10(2)], applied to the conditionally compact neighbourhood Y, there exists a neighbourhood V_3 of z and $k \in \mathbb{N}$ such that if $v \in V_3$, $\lambda_v(Y) \geq k > 0$. By shrinking, we may take $V_3 \subseteq V_2 \cap C$.

Let $u_n \in V_3$. Then $\lambda_{u_n}(Y) \geq k$. Now let a be a real number so that $0 < a \leq \frac{1}{16}k^2$. Since $\|\xi_n\|^2 = \lambda_{u_n}(Y)$ we get

$$\begin{aligned}\|L^{u_n}(F^*F)P_{u_n,1}\| = \|L^{u_n}(F)P_{u_n,1}\|^2 &= \sup_{\|\eta\|=1} \|L^{u_n}(F)\eta\|^2 \\ \geq \|L^{u_n}(F)\frac{\xi_n}{\|\xi_n\|}\|^2 &= \frac{1}{16}\lambda_{u_n}(Y)^2 \geq a. \qquad \square\end{aligned}$$

Now that we have our a, choose $l \in \mathbb{P}$ so that $la > M$. Since $\{u_n\}$ converges k times to z in $G^{(0)}/G$ for every k, there exist l sequences $\{u_n = \gamma_n^{(1)}\}, \{\gamma_n^{(2)}\}, ..., \{\gamma_n^{(l)}\}$ satisfying the two items on page 10.

For each n and $1 \leq i \leq l$, we define the subspace

$$\mathcal{H}_{u_n,i} = \mathcal{H}_{u_n} \cap L^2(E_{u_n} \cap j^{-1}(W_0V_0W_0\gamma_n^{(i)}), \sigma_{u_n}).$$

Let $P_{u_n,i}$ be the projection onto $\mathcal{H}_{u_n,i}$ for $1 \leq i \leq l$.

Lemma 4.5. *The $\mathcal{H}_{u_n,i}$ $(1 \leq i \leq l)$ are invariant under $L^{u_n}(F)$, and the $\mathcal{H}_{u_n,i}$ are eventually pairwise disjoint.*

Proof. Fix $1 \leq i \leq l$. To see that $\mathcal{H}_{u_n,i}$ is invariant under $L^{u_n}(F)$ it suffices to show that $\operatorname{supp}(L^{u_n}(F)\psi) \subseteq j^{-1}(W_0V_0W_0\gamma_n^{(i)})$ for $\psi \in \mathcal{H}_{u_n,i}$ with compact support. Fix $\gamma \in \operatorname{supp}(L^{u_n}(F)\psi)$. Thus

$$0 \neq L^{u_n}(F)\psi(\gamma) = g(r(\gamma))\int_G g(r(\alpha))b(j(\gamma\alpha^{-1}))h(\gamma\alpha^{-1})\psi(\alpha)\ d\lambda_{u_n}(j(\alpha)).$$

For the integral to be non-zero, there must exist $\alpha \in s^{-1}(\{u_n\})$ in the support of the integrand. Then $s(\gamma) = s(\alpha) = u_n$ and, in particular, $j(\gamma) \in G_{u_n} \subseteq G_A$. Also $j(\alpha) \in W_0V_0W_0\gamma_n^{(i)}$.

Suppose, by way of contradiction, that $j(\gamma) \notin \overline{W_1}^7\overline{V_1}\,\overline{W_1}^7\gamma_n^{(i)}$. We have $j(\gamma\alpha^{-1}) \in \operatorname{supp} b \subseteq \overline{W_1}^4\overline{V_1}\,\overline{W_1}^4$. But now $j(\gamma) = j(\gamma\alpha^{-1})j(\alpha) \subseteq \overline{W_1}^4\overline{V_1}\,\overline{W_1}^7\gamma_n^{(i)}$, contradicting that $j(\gamma) \notin \overline{W_1}^7\overline{V_1}\,\overline{W_1}^7\gamma_n^{(i)}$. So $j(\gamma) \in \overline{W_1}^7\overline{V_1}\,\overline{W_1}^7\gamma_n^{(i)}$.

Now suppose, again by way of contradiction, that $j(\gamma) \notin W_0V_0W_0\gamma_n^{(i)}$. Then

$$r(\gamma) = r(j(\gamma)) \in r\big((\overline{W_1}^7\overline{V_1}\,\overline{W_1}^7\gamma_n^{(i)} \setminus W_0V_0W_0\gamma_n^{(i)}) \cap G_A\big) \subseteq r^{-1}(G^{(0)} \setminus \operatorname{supp} g)$$

by (4.4). But now $g(r(\gamma)) = 0$, contradicting that $L^{u_n}(F)\psi(\gamma) \neq 0$. Thus $j(\gamma) \in W_0V_0W_0\gamma_n^{(i)}$. Hence $\mathcal{H}_{u_n,i}$ is invariant under $L^{u_n}(F)$ for $1 \leq i \leq l$.

Next, suppose that $1 \leq i < j \leq l$ and that $\mathcal{H}_{u_n,j}$ and $\mathcal{H}_{u_n,i}$ are not eventually disjoint. Then $j^{-1}(W_0V_0W_0\gamma_n^{(j)})$ and $j^{-1}(W_0V_0W_0\gamma_n^{(i)})$ are not eventually disjoint. So there exists subsequences $\{\gamma_{n_k}^{(i)}\}$, $\{\gamma_{n_k}^{(j)}\}$ of $\{\gamma_n^{(i)}\}$, $\{\gamma_n^{(j)}\}$, and a sequence $\{\alpha_k\} \subseteq G$ such that

$$\alpha_k \in W_0V_0W_0\gamma_{n_k}^{(i)} \cap W_0V_0W_0\gamma_{n_k}^{(j)}.$$

Thus $\alpha_k(\gamma_{n_k}^{(i)})^{-1} \in W_0V_0W_0r(\gamma_{n_k}^{(i)}) \subseteq W_0V_0W_0V_3$ and $\alpha_k(\gamma_{n_k}^{(j)})^{-1} \in W_0V_0W_0V_3$ eventually. So

$$\gamma_{n_k}^{(j)}(\gamma_{n_k}^{(i)})^{-1} = \gamma_{n_k}^{(j)}s(\alpha_k)(\gamma_{n_k}^{(i)})^{-1} = \gamma_{n_k}^{(j)}(\alpha_k)^{-1}\alpha_k(\gamma_{n_k}^{(i)})^{-1} \in V_3^{-1}W_0V_0W_0^2V_0W_0V_3$$

eventually. But $V_3^{-1}W_0V_0W_0^2V_0W_0V_3$ is relatively compact, so $\{\gamma_{n_k}^{(j)}(\gamma_{n_k}^{(i)})^{-1}\}$ has a convergent subsequence. But this contradicts the l-times convergence of $\{u_n\}$. $\square$

Lemma 4.6. *Let F be the function defined at (4.5) and $a > 0$ be as in Lemma 4.4. Suppose that $u_n \in V_3$ and $r(\gamma_n^{(i)}) \in V_3 \cap C$ for $1 \le i \le l$. Then*

$$\|L^{u_n}(F^*F)P_{u_n,i}\| \ge a$$

for $1 \le i \le l$.

Notice that in Lemma 4.4 above, we proved Lemma 4.6 in the special case where $i = 1$; we needed to do the base case $i = 1$ to find the constant a. The proof of Lemma 4.6 is similar to that of Lemma 4.4.

Proof. Let $Y, \mathcal{O}_1, \mathcal{O}_2, c, Y_0, \tilde{Y}, \mathbb{T}_0$ be as in Lemma 4.4. Fix i and $x \in Y\gamma_n^{(i)}$. Then

$$j(c(\gamma_n^{(i)})c(x)^{-1}) = \gamma_n^{(i)}x^{-1} \in r(\gamma_n^{(i)})Y \subseteq CY \subseteq CY_0 \subseteq j(\mathcal{O}_1).$$

There exists a $t \in \mathbb{T}_0$ so that $t \cdot c(\gamma_n^{(i)})c(x)^{-1} \in \mathcal{O}_1$. Just as we defined $\zeta_n : Yu_n \to \mathbb{T}_0$ in Lemma 4.4 we now define $\zeta_n^i : Y\gamma_n^{(i)} \to \mathbb{T}$ by $\zeta_n^i(y) = t_j$ where $j = \min\{k : t_k \cdot c(\gamma_n^{(i)})c(y)^{-1} \in \mathcal{O}_1\}$. Since $r(\gamma_n^{(i)}) \in C$, we have

$$\tilde{Y}\zeta_n^i(x)\gamma_n^{(i)}c(x^{-1}) \subseteq \tilde{Y}r(\gamma_n^{(i)})\tilde{Y} \subseteq \tilde{Y}j^{-1}(CY) \subseteq \mathcal{O}_2.$$

Let χ_n^i be the characteristic function of $Y\gamma_n^{(i)}$ and define $\xi_n^i : E \to \mathbb{C}$ by

$$\xi_n^i(\gamma) = t(\gamma)\zeta_n^i(j(\gamma))\chi_n^i(j(\gamma)).$$

Since all of the functions involved in defining ξ_n^i are Borel, so is ξ_n^i. It is clear that ξ_n^i is bounded, $\mathbb{T}$-invariant and has compact support in $j^{-1}(\overline{Y\gamma_n^{(i)}})$. Since $s(\gamma_n^{(i)}) = u_n$ we have $\operatorname{supp}\xi_n^i \subseteq E_{u_n} \cap j^{-1}(Y\gamma_n^{(i)}) \subseteq E_{u_n} \cap j^{-1}(W_0V_0W_0\gamma_n^{(i)})$. Thus $\xi_n^i \in \mathcal{H}_{u_n,i}$.

Fix $\gamma \in j^{-1}(Y\gamma_n^{(i)})$. If $\alpha \in j^{-1}(Y\gamma_n^{(i)})$ then $j(\gamma\alpha^{-1}) \in Yr(\gamma_n^{(i)})Y \subseteq W_0V_0W_0$ and hence $b(j(\gamma\alpha^{-1})) = 1$. The support of ξ_n^i is $Y\gamma_n^{(i)}$, so the same calculation as done on page 13 gives

$$L^{u_n}(F)(\xi_n^i)(\gamma) = g(r(\gamma))\int_{Y\gamma_n^{(i)}} g(r(y))h(\gamma c(y)^{-1})\xi_n^i(c(y))\,d\lambda_{u_n}(y)$$

which, since $r(\gamma_n^{(i)}) \in V_2$ implies $r(Y\gamma_n^{(i)}) \subseteq U$ and since g is identically one on U, is

$$\begin{aligned} &= \int_{Y\gamma_n^{(i)}} h(\gamma c(y)^{-1})\xi_n^i(c(y))\,d\lambda_{u_n}(y) \\ &= \int_{Y\gamma_n^{(i)}} h(\gamma\zeta_n^i(y)c(y)^{-1})\,d\lambda_{u_n}(y). \end{aligned}$$

But $\gamma\zeta_n^i(y)c(y)^{-1} \in \mathcal{O}_2$, so $\operatorname{Re}(h(\gamma\zeta_n^i(y)c(y)^{-1})) > \frac{1}{4}$ and

$$\operatorname{Re}\big(L^{u_n}(F)(\xi_n^i)(\gamma)\big) \ge \frac{1}{4}\lambda_{u_n}(Y\gamma_n^{(i)}) = \frac{1}{4}\lambda_{r(\gamma_n^{(i)})}(Y).$$

Since $r(\gamma_n^{(i)}) \in V_3 \cap C$ by assumption, the same calculation as at the end of the proof of Lemma 4.4 gives $\|L^{u_n}(F^*F)P_{u_n,i}\| \ge a$. ∎

To push F^*F into the Pedersen ideal, let $q \in C_c(0,\infty)$ be any function satisfying

$$q(t) = \begin{cases} 0, & \text{if } t < \frac{a}{3}, \\ 2t - \dfrac{2a}{3}, & \text{if } \frac{a}{3} \le t < \frac{2a}{3}, \\ t, & \text{if } \frac{2a}{3} \le t \le \|F^*F\|. \end{cases}$$

Set $d := q(F^*F)$. We will show that $\operatorname{tr}(L^{u_n}(d)) \ge la > M$ eventually.

Fix $u_n \in V_3$ such that $r(\gamma_n^{(i)}) \in V_3 \cap C$ for $1 \le i \le l$. By Lemma 4.6, each $L^{u_n}(F^*F)P_{u_n,i}$ is positive with an eigenvalue at least as large as a, and by choice of q each $q(L^{u_n}(F^*F)P_{u_n,i})$ is positive with norm at least as large as a.

We claim that

$$q(L^{u_n}(F^*F)P_{u_n,i}) = q(L^{u_n}(F^*F))P_{u_n,i}.$$

To see the claim, let p be a polynomial that vanishes at 0. Since $L^{u_n}(F^*F)$ leaves $\mathcal{H}_{u_n,i}$ invariant, $L^{u_n}(F^*F)$ and $P_{u_n,i}$ commute. If we plug the operator $L^{u_n}(F^*F)P_{u_n,i}$ into p and simplify, we see that $p(L^{u_n}(F^*F)P_{u_n,i}) = p(L^{u_n}(F^*F))P_{u_n,i}$. Because q can be uniformly approximated by polynomials p, each vanishing at 0, the claim follows. Now

$$\begin{aligned} \|L^{u_n}(d)\| = \|(L^{u_n}(q(F^*F))\| &\ge \sum_{i=1}^{l} \|L^{u_n}(q(F^*F))P_{u_n,i}\| = \sum_{i=1}^{l} \|q(L^{u_n}(F^*F))P_{u_n,i}\| \\ &= \sum_{i=1}^{l} \|q(L^{u_n}(F^*F)P_{u_n,i})\| \ge la > M \end{aligned}$$

by the choice of l. Thus $\operatorname{tr}(L^{u_n}(d)) > M$, and since M was arbitrary d is not a bounded-trace element. But d is an element of the Pedersen ideal of $C^*(E;G,\lambda)$, so $C^*(E;G,\lambda)$ does not have bounded trace. This completes the proof of Theorem 4.3.

5. The twisted groupoid C^*-algebras that are Fell Algebras

Recall from [19] that a groupoid G is *proper* if the map $\pi : G \to G^{(0)} \times G^{(0)}$, defined by $\pi(\gamma) = (r(\gamma), s(\gamma))$ for $\gamma \in G$, is a proper map. A subset U of $G^{(0)}$ is *wandering* if $G|_U = \pi^{-1}(U \times U)$ is relatively compact. Thus G is proper if and only if every compact subset of $G^{(0)}$ is wandering. A groupoid G where each unit has a wandering neighbourhood is called *Cartan* [4, Definition 7.3].

The following lemma illustrates the relationship between a Cartan groupoid and 2-times convergence in the orbit space of the groupoid; it is similar to one direction of [2, Lemma 2.3]. Lemma 5.1 will be used in Example 7.1 below.

Lemma 5.1. *Let G be a topological groupoid. If there exists a sequence $\{u_n\} \subseteq G^{(0)}$ which converges 2-times in $G^{(0)}/G$ to $z \in G^{(0)}$, then G is not Cartan.*

Proof. We argue by contradiction. Suppose that $\{u_n\} \subseteq G^{(0)}$ converges 2-times in $G^{(0)}/G$ to z and that G is Cartan. Let U be a wandering neighbourhood of z in $G^{(0)}$, so that $G|_U$ is relatively compact. There exist sequences $\{\gamma_n^{(1)}\}$, $\{\gamma_n^{(2)}\} \subseteq G$ such that $r(\gamma_n^{(i)}) \to z$, $s(\gamma_n^{(i)}) = u_n$ for $i = 1, 2$ and $\gamma_n^{(2)}(\gamma_n^{(1)})^{-1} \to \infty$ as $n \to \infty$. Thus $\gamma_n^{(2)}, \gamma_n^{(1)} \in G|_U$ eventually, and hence $\gamma_n^{(2)}(\gamma_n^{(1)})^{-1} \in \overline{G|_U G|_U}$, eventually. But $\overline{G|_U G|_U}$ is compact, contradicting that $\gamma_n^{(2)}(\gamma_n^{(1)})^{-1} \to \infty$ as $n \to \infty$.

□

Theorem 5.2. *Let E be a second-countable, locally compact, Hausdorff, $\mathbb{T}$-groupoid such that $G := E/\mathbb{T}$ is a principal groupoid with Haar system λ. The following are equivalent:*

1. *the twisted groupoid C^*-algebra $C^*(E; G, \lambda)$ is a Fell algebra;*
2. *G is Cartan;*
3. *$C^*(G)$ is a Fell algebra.*

Proof. Since G is principal, (2) and (3) are equivalent by [4, Theorem 7.9]; we will now prove the equivalence of (1) and (2).

Suppose that G is Cartan. Fix an irreducible representation ρ of $C^*(E; G, \lambda)$; we will show that ρ satisfies Fell's condition. Since G is Cartan, $G^{(0)}/G$ is T_1 by [4, Lemma 7.4]. So by Proposition 3.2, ρ is unitarily equivalent to $L^u(E; G)$ for some $u \in G^{(0)}$. It suffices to show $L^u(E; G)$ satisfies Fell's condition. Let U_0 be a wandering neighbourhood of u in $G^{(0)}$ and $U = r(s^{-1}(U_0))$ its saturation. Since G has a Haar system, r is open and hence U is open. By [4, Lemma 7.8], $G|_U$ is a proper groupoid, so by [19, Theorem 4.2], $C^*(E|_U; G|_U, \lambda)$ has continuous trace. By Lemma 3.1, the inclusion $k : C_c(E|_U; G|_U) \to C_c(E; G)$ induces an isometric isomorphism k of $C^*(E|_U; G|_U, \lambda)$ onto an ideal I of $C^*(E; G, \lambda)$. Thus I has continuous trace and $L^u(E; G)|_I = L^u(E|_U; G|_u) \circ k^{-1}$. Since I has continuous trace, $L^u(E; G)|_I$ satisfies Fell's condition in $\hat{I}$, and hence $L^u(E; G)$ satisfies Fell's condition in $C^*(E; G, \lambda)^\wedge$. Thus ρ satisfies Fell's condition in $C^*(E; G, \lambda)^\wedge$ as well and $C^*(E; G, \lambda)$ is a Fell algebra.

Conversely, suppose that $C^*(E; G, \lambda)$ is a Fell algebra. Fix $u \in G^{(0)}$; we will show that u has a wandering neighbourhood in $G^{(0)}$. Since $C^*(E; G, \lambda)$ is liminal, $G^{(0)}/G$ is T_1 by Proposition 3.3 and $L : G^{(0)}/G \to C^*(E; G, \lambda)^\wedge$, $[u] \mapsto [L^u]$ is a homeomorphism by Proposition 3.2. By [3, Corollary 3.4], $[L^u]$ has an open Hausdorff neighbourhood O in $C^*(E; G, \lambda)^\wedge$. Let $q : G^{(0)} \to G^{(0)}/G$ the quotient map and set $U = q^{-1}(L^{-1}(O))$. Then U is an open saturated subset of $G^{(0)}$ and $C^*(E|_U; G|_U, \lambda)$ is isomorphic to an ideal I of $C^*(E; G, \lambda)$ with spectrum O. Thus $C^*(E|_U; G|_U, \lambda)$ has continuous trace (because I has) and hence $G|_U$ is a proper groupoid by [19, Theorem 4.3]. So any relatively compact neighbourhood contained in U is a wandering neighbourhood of u in $G^{(0)}$. Thus G is Cartan. □

6. Groupoids with abelian isotropy groups.

Throughout this section $\mathcal{G}$ is a second-countable, locally compact, Hausdorff groupoid with Haar system λ. The change in notation from G to $\mathcal{G}$ is to emphasize that we are no longer assuming that the groupoid $\mathcal{G}$ is principal. Let $A_u = \{\gamma \in \mathcal{G} : r(\gamma) = s(\gamma) = u\}$ be the isotropy group at $u \in \mathcal{G}^{(0)}$ and let $\mathcal{A} = \{\gamma \in \mathcal{G} : r(\gamma) = s(\gamma)\}$ be the isotropy groupoid; we also assume throughout this section that the isotropy groups are abelian and vary continuously, that is, that the map $u \mapsto A_u$ from $\mathcal{G}^{(0)}$ to the space of closed subsets of $\mathcal{G}^{(0)}$, is continuous in the Fell topology. The isotropy groupoid acts on the left and right of $\mathcal{G}$ and the quotient $\mathcal{R} := \mathcal{G}/\mathcal{A}$ is a principal groupoid. The main results of this section, Theorems 6.4 and 6.5, say that $C^*(\mathcal{G}, \lambda)$ has bounded trace if and only if $\mathcal{R}$ is an integrable groupoid, and that $C^*(\mathcal{G}, \lambda)$ is a Fell algebra if and only if $\mathcal{R}$ is a Cartan groupoid. Once again, our proofs are modeled after the analogous result [21, Theorem 1.1] for groupoid C^*-algebras with continuous trace.

Since the isotropy groups vary continuously, $\mathcal{A}$ has a Haar system β [27, Lemmas 1.1 and 1.2]. Write $\hat{\mathcal{A}}$ for the spectrum of $C^*(\mathcal{A},\beta)$. Then $\mathcal{R}$ acts on the right of $\hat{\mathcal{A}}$ (see (6.1) and (6.2) below). In [21] Muhly, Renault, and Williams show that if $\hat{\mathcal{A}}/\mathcal{R}$ is Hausdorff, then $C^*(\mathcal{G},\lambda)$ is isomorphic to a particular twisted groupoid C^*-algebra [21, Proposition 4.5]. They then apply the characterization of when twisted groupoid C^*-algebras have continuous trace from [20] to prove [21, Theorem 1.1].

Our strategy is similar. We prove in Lemma 6.1 that $\mathcal{G}^{(0)}/\mathcal{G}$ is T_1 if and only if $\hat{\mathcal{A}}/\mathcal{R}$ is T_1. This allows us to show that the isomorphism of [21, Proposition 4.5] holds even if $\hat{\mathcal{A}}/\mathcal{R}$ is only T_1. Then we use the isomorphism and our characterizations in Theorems 4.3 and 5.2 of when twisted groupoid C^*-algebras have bounded trace or are Fell algebras to get results for $C^*(\mathcal{G},\lambda)$.

We need some background before we can proceed to Lemma 6.1. Since $C^*(\mathcal{A},\beta)$ is a separable commutative C^*-algebra, the discussion on [21, p. 3630] shows that

$$\hat{\mathcal{A}} = \{(\chi,u) : u \in \mathcal{G}^{(0)}, \chi \in \hat{A}_u\} \tag{6.1}$$

where $(\chi,u)(f) = \int_{A_u} \chi(a)f(a)\,d\beta^u(a)$ for $f \in C_c(\mathcal{A})$. Proposition 3.3 of [21] describes criteria for convergence in $\hat{\mathcal{A}}$: $(\chi_n,u_n) \to (\chi,u)$ in $\hat{\mathcal{A}}$ if and only if (1) $u_n \to u$ in $\mathcal{A}^{(0)}(=\mathcal{G}^{(0)})$, and (2) if $a_n \in A_{u_n}$, $a \in A_u$ and $a_n \to a$ in $\mathcal{A}$, then $\chi_n(a_n) \to \chi(a)$.

If $\chi \in \hat{A}_u$ and $\gamma \in \mathcal{G}$ with $r(\gamma) = u$, then $\chi\cdot\gamma$ is the character of $A_{s(\gamma)}$ defined by $\chi\cdot\gamma(a) = \chi(\gamma^{-1}a\gamma)$. Note that $\chi\cdot\gamma$ depends only on $\dot{\gamma}$. There is a groupoid action of $\mathcal{R}$ (and $\mathcal{G}$) on the right of $\hat{\mathcal{A}}$ via

$$(\chi,u)\cdot\dot{\gamma} = (\chi\cdot\gamma, s(\gamma)) \tag{6.2}$$

for $\gamma \in \mathcal{G}$ with $r(\gamma) = u$.

Lemma 6.1. *Suppose that $\mathcal{G}$ is a second-countable, locally compact, Hausdorff groupoid with Haar system. Also assume that the isotropy groups are abelian and vary continuously. Then $\mathcal{G}^{(0)}/\mathcal{G}$ is T_1 if and only if $\hat{\mathcal{A}}/\mathcal{R}$ is T_1.*

Proof. First suppose that $\mathcal{G}^{(0)}/\mathcal{G}$ is T_1. Fix $(\rho,v) \in \hat{\mathcal{A}}$. It suffices to show that $[(\rho,v)]$ is closed. Let $(\chi_n,u_n) \in [(\rho,v)]$ and suppose that $(\chi_n,u_n) \to (\chi,u)$ in $\hat{\mathcal{A}}$. Thus there exists $\gamma \in \mathcal{G}$ with $s(\gamma) = u$ and $r(\gamma) = v$, and, for each n, there exists $\gamma_n \in \mathcal{G}$ with $s(\gamma_n) = u_n$, $r(\gamma_n) = v$ such that $(\chi_n,u_n) = (\rho\cdot\gamma_n,u_n) = (\rho,v)\cdot\dot{\gamma}_n$. Note $u \in [v]$ since $u_n \in [v]$ and $[v]$ is closed by assumption, and that $\gamma,\gamma_n \in \mathcal{G}|_{[v]}$.

Since $\mathcal{G}$ has a Haar system, r and s are open maps [25, Proposition 2.4] and this puts us in the setting of [21]. Since $\mathcal{G}|_{[v]}$ is a transitive groupoid, the map $\pi : \mathcal{G}|_{[v]} \to [v]\times[v]$, $\pi(\alpha) = (r(\alpha),s(\alpha))$ is open by [21, Theorems 2.2A and 2.2B]. Since $\pi(\gamma_n) = (v,u_n) \to (v,u) = \pi(\gamma)$ and π is open, there exists a subsequence $\{\gamma_{n_k}\}$ and a sequence $\{\eta_k\} \subseteq \mathcal{G}$ such that $\pi(\gamma_{n_k}) = \pi(\eta_k)$ and $\eta_k \to \gamma$ in $\mathcal{G}|_{[v]}$ (see, for example, [30, Proposition 1.15]). Thus $\eta_k \to \gamma$ in $\mathcal{G}$ as well. Note that $\dot{\gamma}_{n_k} = \dot{\eta}_k$.

Fix a sequence $\{a_k\}$ with $a_k \in A_{u_{n_k}}$ such that $a_k \to a$ in $\hat{\mathcal{A}}$. Then by the continuity of multiplication,

$$\chi_k(a_k) = (\rho\cdot\dot{\gamma}_{n_k})(a_k) = \rho(\eta_k^{-1}a_k\eta_k) \to \rho(\gamma^{-1}a\gamma) = (\rho\cdot\dot{\gamma})(a).$$

Thus $\{(\chi_{n_k}, u_{n_k})\}$ converges to both (χ, u) and $(\rho \cdot \dot\gamma, u)$ in $\hat{\mathcal{A}}$. Since $\hat{\mathcal{A}}$ is Hausdorff we have

$$(\chi, u) = (\rho \cdot \dot\gamma, u) = (\rho, v) \cdot \dot\gamma \in [(\rho, v)].$$

So $[(\rho, v)]$ is closed. Hence $\hat{\mathcal{A}}/\mathcal{R}$ is T_1.

For the converse, first consider $\phi : \mathcal{G}^{(0)} \to \hat{\mathcal{A}}/\mathcal{R}$ defined by $\phi(u) = [(1_u, u)]$, where 1_u is the trivial character $a \mapsto 1$ for $a \in A_u$. If $u_n \to u$ in $\mathcal{G}^{(0)}$ then, using the convergence criteria for sequences in $\hat{\mathcal{A}}$ of [21, Proposition 3.3], it is clear that $(1_{u_n}, u_n) \to (1_u, u)$ in $\hat{\mathcal{A}}$, so that ϕ is continuous. Now suppose that $\phi(u) = \phi(v)$. Then there exists $\gamma \in \mathcal{G}$ with $r(\gamma) = u$ such that $(1_u, u) \cdot \dot\gamma = (1_v, v)$. Thus $v = s(\gamma)$ and hence $[u] = [v]$. So ϕ induces a continuous injection $\phi : \mathcal{G}^{(0)}/\mathcal{G} \to \hat{\mathcal{A}}/\mathcal{R}$. It follows that $\mathcal{G}^{(0)}/\mathcal{G}$ is T_1 if $\hat{\mathcal{A}}/\mathcal{R}$ is. □

In [21], Muhly, Renault, and Williams define a groupoid $\hat{\mathcal{A}} \rtimes \mathcal{R}$ as follows. As a set $\hat{\mathcal{A}} \rtimes \mathcal{R} = \{(\chi, r(\gamma), \dot\gamma) \in \hat{\mathcal{A}} \times \mathcal{R}\}$, but an element $(\chi, r(\gamma), \dot\gamma)$ is abbreviated to just $(\chi, \dot\gamma)$; the topology on $\hat{\mathcal{A}} \rtimes \mathcal{R}$ is the product topology. The unit space is $\hat{\mathcal{A}}$ with range and source maps

$$r((\chi, \dot\gamma)) = (\chi, r(\gamma)) \text{ and } s((\chi, \dot\gamma)) = (\chi \cdot \gamma, s(\gamma)).$$

The multiplication and inverse in $\hat{\mathcal{A}} \rtimes \mathcal{R}$ is given by

$$(\chi, \dot\gamma)(\chi \cdot \dot\gamma, \dot\alpha) = (\chi, \dot\gamma\dot\alpha) \text{ and } (\chi, \dot\gamma)^{-1} = (\chi \cdot \dot\gamma, \dot\gamma^{-1}).$$

It is straightforward to see that $\hat{\mathcal{A}} \rtimes \mathcal{R}$ is principal. Note that $\mathcal{R}$ is proper if and only if $\hat{\mathcal{A}} \rtimes \mathcal{R}$ is proper; similarly one is Cartan or integrable if and only if the other is:

Lemma 6.2. *Suppose that $\mathcal{G}$ is a second-countable, locally compact, Hausdorff groupoid with abelian isotropy. Also assume that the isotropy groupoid $\mathcal{A}$ has a Haar system.*

1. *If $\mathcal{G}$ has a Haar system, then $\mathcal{R}$ and $\hat{\mathcal{A}} \rtimes \mathcal{R}$ have Haar systems α and $\delta \times \alpha$, respectively; and with respect to these Haar systems, $\mathcal{R}$ is integrable if and only if $\hat{\mathcal{A}} \rtimes \mathcal{R}$ is integrable.*
2. *$\mathcal{R}$ is Cartan if and only if $\hat{\mathcal{A}} \rtimes \mathcal{R}$ is Cartan.*

Proof. (1) Since $\mathcal{G}$ and $\mathcal{A}$ have Haar systems, $\mathcal{R}$ has a Haar system α by [21, Lemma 4.2]. It is straightforward to check that if $(\chi, u) \in \hat{\mathcal{A}}$ and $\delta_{(\chi,u)}$ is point-mass measure, then $\delta \times \alpha^{(\chi,u)} := \delta_{(\chi,u)} \times \alpha^u$ gives a Haar system on $\hat{\mathcal{A}} \rtimes \mathcal{R}$.

Suppose $\mathcal{R}$ is integrable. Fix a compact subset K in $(\hat{\mathcal{A}} \rtimes \mathcal{R})^{(0)} = \hat{\mathcal{A}}$. Let $p_2 : \hat{\mathcal{A}} \rtimes \mathcal{R} \to \mathcal{R}$ be the projection onto the second coordinate; note that $p_2(K)$ is a compact subset of $R^{(0)} = \mathcal{G}^{(0)}$. We have

$$\begin{aligned}(\delta \times \alpha)^{(\chi,u)}(s^{-1}(K)) &= \delta_{(\chi,u)} \times \alpha^u\big(\{(\eta, \dot\gamma) \in \hat{\mathcal{A}} \rtimes \mathcal{R} : (\eta \cdot \gamma, s(\gamma)) \in K\}\big)\\ &= \alpha^u\big(\{\dot\gamma \in \mathcal{R} : r(\gamma) = u, (\chi \cdot \gamma, s(\gamma)) \in K\}\big)\\ &\le \alpha^u\big(\{\dot\gamma \in \mathcal{R} : s(\gamma) \in p_2(K)\}\big)\\ &= \alpha^u\big(s^{-1}(p_2(K))\big).\end{aligned}$$

Since $\mathcal{R}$ is integrable, this gives

$$\sup_{(\chi,u)\in K} \big\{(\delta \times \alpha)^{(\chi,u)}(s^{-1}(K))\big\} \le \sup_{u \in p_2(K)} \big\{\alpha^u(s^{-1}(p_2(K))\})\big\} < \infty.$$

So $\hat{\mathcal{A}} \rtimes \mathcal{R}$ is integrable.

Conversely, suppose that $\hat{\mathcal{A}} \rtimes \mathcal{R}$ is integrable. Fix a compact set L in $\mathcal{R}^{(0)} = \mathcal{G}^{(0)}$. For each $u \in L$, let 1_u be the trivial character of A_u, so that $a \mapsto 1$ for all $a \in A_u$. Set $\tilde{L} = \{(1_u, u) : u \in L\} \subseteq \hat{\mathcal{A}}$. We claim that $\tilde{L}$ is a compact subset of $(\hat{\mathcal{A}} \rtimes \mathcal{R})^{(0)} = \hat{\mathcal{A}}$. To see this, let $\{(1_{v_n}, v_n)\}$ be a sequence in $\tilde{L}$. Then $\{v_n\}$ is a sequence in L and hence has a convergent subsequence $v_{n_k} \to v \in L$. Using the convergence criteria for sequences in $\hat{\mathcal{A}}$ of [21, Proposition 3.3] it is clear that $(1_{v_{n_k}}, v_{n_k}) \to (1_v, v)$ in $\hat{\mathcal{A}}$. Thus $\tilde{L}$ is compact in $\hat{\mathcal{A}}$.

Now note that $s((1_u, \dot{\gamma})) = (1_u, s(\gamma))$, so that $s((1_u, \dot{\gamma})) \in \tilde{L}$ if and only if $s(\dot{\gamma}) \in L$. Thus

$$\sup_{u \in L} \{\alpha^u(s^{-1}(L))\} \leq \sup_{(1_u,u) \in \tilde{L}} \{(\delta \times \alpha)^{(1_u,u)}(s^{-1}(\tilde{L}))\} < \infty$$

because $\hat{\mathcal{A}} \rtimes \mathcal{R}$ is integrable and $\tilde{L}$ is a compact subset of its unit space. So $\mathcal{R}$ is integrable.

(2) First suppose that $\mathcal{R}$ is Cartan. Fix $(\chi, u) \in \hat{\mathcal{A}} = (\hat{\mathcal{A}} \rtimes \mathcal{R})^{(0)}$. Let K be a relatively compact, wandering neighbourhood of u in $\mathcal{R}^{(0)}$. Let $p_1 : \hat{\mathcal{A}} \rtimes \mathcal{R} \to \mathcal{A}$, $p_2 : \hat{\mathcal{A}} \rtimes \mathcal{R} \to \mathcal{R}$ be the projections onto the first and second coordinate, respectively. Let N be a relatively compact neighbourhood of (χ, u) in $\hat{\mathcal{A}}$ such that $p_2(N) = K$. Let $\{(\eta_n, \dot{\gamma}_n)\}$ be a sequence in $\pi^{-1}(N \times N) = \{(\chi, \dot{\gamma}) : (\chi, r(\gamma)) \in N, (\chi \cdot \dot{\gamma}, s(\gamma)) \in N\}$. Then $\{\dot{\gamma}_n\} \subseteq \pi^{-1}(p_2(N) \times p_2(N)) = \pi^{-1}(K \times K)$, hence has a convergent subsequence $\{\dot{\gamma}_{n_k}\}$. Note $\{\eta_{n_k}\} \subseteq p_1(N)$, a relatively compact set. So there exists a convergent subsequence $\{\eta_{n_{k_i}}\}$. So $\{(\eta_{n_{k_i}} \dot{\gamma}_{n_{k_i}}\}$ is a convergent subsequence of $\{(\eta_n, \dot{\gamma}_n)\}$. Thus $\pi^{-1}(N \times N)$ is relatively compact. Hence $\hat{\mathcal{A}} \rtimes \mathcal{R}$ is Cartan.

Conversely, suppose that $\hat{\mathcal{A}} \rtimes \mathcal{R}$ is Cartan. Fix $u \in \mathcal{R}^{(0)}$. There exists a wandering neighbourhood N of $(1_u, u)$ in $\hat{\mathcal{A}}$. Let $K = p_2(N)$; then K is a neighbourhood of u. Let $\{\dot{\gamma}_n\} \subseteq \pi^{-1}(K \times K)$. For each n there exists η_n such that $(\eta_n, \dot{\gamma}_n) \in \pi^{-1}(N \times N)$. But $\pi^{-1}(N \times N)$ is relatively compact, so $\{(\eta_n, \dot{\gamma}_n)\}$ has a convergent subsequence $\{(\eta_{n_k}, \dot{\gamma}_{n_k})\}$. Thus $\{\dot{\gamma}_{n_k}\}$ is a convergent subsequence of $\{\dot{\gamma}_n\}$. Thus $\pi^{-1}(K \times K)$ is relatively compact. Hence $\mathcal{R}$ is Cartan. □

We will now briefly describe the $\mathbb{T}$-groupoid $\mathcal{D}$ of [21, §4]. There

$$\mathcal{D} := \{(\chi, z, \gamma) : \chi \in \hat{A}_{r(\gamma)}, z \in \mathbb{T}, \gamma \in \mathcal{G}\}/\sim, \tag{6.3}$$

where $(\chi, \chi(a)z, \gamma) \sim (\chi, z, a \cdot \gamma)$; the unit space is $\hat{\mathcal{A}}$ with

$$r([(\chi, z, \gamma)]) = (\chi, r(\gamma)) \quad \text{and} \quad s([(\chi, z, \gamma)]) = (\chi \cdot \gamma, s(\gamma))$$

and multiplication and inverse

$$[(\chi, z, \gamma)][(\chi \cdot \gamma, z', \gamma')] = [(\chi, zz', \gamma\gamma')] \quad \text{and} \quad [(\chi, z, \gamma)]^{-1} = [(\chi \cdot \gamma, \bar{z}, \gamma^{-1})].$$

That $\mathcal{D}$ is indeed a $\mathbb{T}$-groupoid over $\hat{\mathcal{A}} \rtimes \mathcal{R}$ is established on [21, p. 3636].

Proposition 4.5 of [21] says that if $\hat{\mathcal{A}}/\mathcal{R}$ is Hausdorff, then $C^*(\mathcal{G})$ and $C^*(\mathcal{D}; \hat{\mathcal{A}} \rtimes \mathcal{R}, \delta \times \alpha)$ are isomorphic. We now establish that the given proof works almost as is written even if $\hat{\mathcal{A}}/\mathcal{R}$ is only T_1. Proposition 4.5 of [21] uses the Hausdorff assumption in three places. The first use is in Lemma 4.8 to establish that the $\mathcal{G}$-orbits in $\mathcal{G}^{(0)}$ are closed; so assuming $\hat{\mathcal{A}}/\mathcal{R}$ is T_1 suffices by Lemma 6.1. The second use is to establish again that the $\mathcal{G}$-orbits are closed in $\mathcal{G}^{(0)}$ so that [21, Lemma 2.11] applies. The third use is to establish that every irreducible representation of $C^*(\mathcal{D}; \hat{\mathcal{A}} \rtimes \mathcal{R}, \delta \times \alpha)$ is of the form $[L^{(\chi,u)}]$; here we

note that $(\hat{\mathcal{A}} \rtimes \mathcal{R})^{(0)}/(\hat{\mathcal{A}} \rtimes \mathcal{R})$ is homeomorphic to $\hat{\mathcal{A}}/\mathcal{R}$, so we can use Proposition 3.2 for this if $\hat{\mathcal{A}}/\mathcal{R}$ is T_1. Thus we have:

Proposition 6.3. *Suppose $\mathcal{G}$ is a second-countable, locally compact, Hausdorff groupoid with Haar system λ. Also suppose that the isotropy groups of $\mathcal{G}$ are abelian and vary continuously. If $\mathcal{G}^{(0)}/\mathcal{G}$ is T_1, then $C^*(\mathcal{G}, \lambda)$ and $C^*(\mathcal{D}; \hat{\mathcal{A}} \rtimes \mathcal{R}, \delta \times \alpha)$ are isomorphic.*

Theorem 6.4. *Suppose that $\mathcal{G}$ is a second-countable, locally compact, Hausdorff groupoid with Haar system λ. Also suppose that the isotropy groups of $\mathcal{G}$ are abelian and vary continuously. Let $\mathcal{A}$ be the isotropy groupoid. The following are equivalent:*

(1) *$C^*(\mathcal{G}, \lambda)$ has bounded trace;*
(2) *$\mathcal{R} := \mathcal{G}/\mathcal{A}$ is integrable;*
(3) *$C^*(\mathcal{R})$ has bounded trace.*

Proof. Since $\mathcal{R}$ is principal, the equivalence of (2) and (3) is [6, Theorem 4.4]; we will now prove the equivalence of (1) and (2). Note that the isotropy groups vary continuously if and only if the isotropy groupoid $\mathcal{A}$ has a Haar system by [27, Lemmas 1.1 and 1.2].

First suppose that $C^*(\mathcal{G}, \lambda)$ has bounded trace. Then $C^*(\mathcal{G}, \lambda)$ is liminal and hence the orbits of $\mathcal{G}$ are closed by [5, Theorem 6.1]. Now $C^*(\mathcal{G}, \lambda)$ and $C^*(\mathcal{D}; \hat{\mathcal{A}} \rtimes \mathcal{R}, \delta \times \alpha)$ are isomorphic by Proposition 6.3. Thus $C^*(\mathcal{D}; \hat{\mathcal{A}} \rtimes \mathcal{R}, \delta \times \alpha)$ has bounded trace as well. Thus $\hat{\mathcal{A}} \rtimes \mathcal{R}$ is integrable by Theorem 4.3, and hence $\mathcal{R}$ is integrable by Lemma 6.2(1).

Conversely, suppose $\mathcal{R}$ is integrable. Then the orbits in $\mathcal{R}$ are closed by [6, Lemma 3.9] and Lemma 4.1. Since $\mathcal{R}$ and $\mathcal{G}$ have the same orbit space, orbits are closed in $\mathcal{G}$. By Proposition 6.3, $C^*(G, \lambda)$ and $C^*(\mathcal{D}; \hat{\mathcal{A}} \rtimes \mathcal{R}, \alpha)$ are isomorphic. Since $\mathcal{R}$ is integrable, $\hat{\mathcal{A}} \rtimes \mathcal{R}$ is integrable by Lemma 6.2(1). Thus $C^*(\mathcal{D}; \hat{\mathcal{A}} \rtimes \mathcal{R}, \alpha)$, and hence $C^*(\mathcal{G}, \lambda)$, has bounded trace by Theorem 4.3. □

Theorem 6.5. *Suppose that $\mathcal{G}$ is a second-countable, locally compact, Hausdorff groupoid with Haar system λ. Also suppose that the isotropy groups of $\mathcal{G}$ are abelian and vary continuously. Let $\mathcal{A}$ be the isotropy groupoid. The following are equivalent:*

(1) *$C^*(\mathcal{G}, \lambda)$ is a Fell algebra;*
(2) *$\mathcal{R} := \mathcal{G}/\mathcal{A}$ is Cartan;*
(3) *$C^*(\mathcal{R})$ is a Fell algebra.*

Proof. Since $\mathcal{R}$ is principal, the equivalence of (2) and (3) is [4, Theorem 7.9]. The proof of the equivalence of (1) and (2) is similar to the proof of Theorem 6.4, using Lemma 6.2(2), Theorem 5.2 and [4, Lemma 4.7] in place of Lemma 6.2(1),Theorem 4.3 and [6, Lemma 3.9], respectively. □

7. Examples

Our examples use groupoids constructed from directed graphs. We start with some background. Let $E = (E^0, E^1, r, s)$ be a directed graph. Thus E^0 and E^1 are countable sets of vertices and edges, respectively, and $r, s : E^1 \to E^0$ are the range and source map, respectively. For $e \in E^1$, call $s(e)$ the source of e and $r(e)$ the range of e. A directed graph E is row-finite if $r^{-1}(v)$ is finite for every $v \in E^0$. A finite path is a finite sequence $\alpha = \alpha_1\alpha_2 \cdots \alpha_k$ of edges $\alpha_i \in E^1$ with $s(\alpha_j) = r(\alpha_{j+1})$ for $1 \leq j \leq k-1$; write $s(\alpha) = s(\alpha_k)$ and $r(\alpha) = r(\alpha_1)$, and call $|\alpha| := k$ the length of α. An infinite

path $x = x_1x_2\cdots$ is defined similarly, although $s(x)$ remains undefined. Let E^* and E^∞ denote the set of all finite paths and infinite paths in E respectively. If $\alpha = \alpha_1\cdots\alpha_k$ and $\beta = \beta_1\cdots\beta_j$ are finite paths with $s(\alpha) = r(\beta)$, then $\alpha\beta$ is the path $\alpha_1\cdots\alpha_k\beta_1\cdots\beta_j$. When $x \in E^\infty$ with $s(\alpha) = r(x)$ define αx similarly. A cycle is a finite path α of non-zero length such that $s(\alpha) = r(\alpha)$. By [18, Corollary 2.2], the cylinder sets

$$Z(\alpha) := \{x \in E^\infty : x_1 = \alpha_1, \dots, x_{|\alpha|} = \alpha_{|\alpha|}\},$$

parameterized by $\alpha \in E^*$, form a basis of compact, open sets for a locally compact, σ-compact, totally disconnected, Hausdorff topology on E^∞.

In [18], Kumjian, Pask, Raeburn and Renault built a groupoid $\mathcal{G}_E$, called the path groupoid, from a row-finite directed graph E as follows. Two paths $x, y \in E^\infty$ are shift equivalent with lag $k \in \mathbb{Z}$ (written $x \sim_k y$) if there exists $N \in \mathbb{N}$ such that $x_i = y_{i+k}$ for all $i \geq N$. Then the groupoid is

$$\mathcal{G}_E := \{(x, k, y) \in E^\infty \times \mathbb{Z} \times E^\infty : x \sim_k y\}.$$

with composable pairs

$$\mathcal{G}_E^{(2)} := \{\big((x, k, y), (y, l, z)\big) : (x, k, y), (y, l, z) \in \mathcal{G}_E\},$$

and composition and inverse given by

$$(x, k, y) \cdot (y, l, z) := (x, k + l, z) \quad \text{and} \quad (x, k, y)^{-1} := (y, -k, x).$$

For each $\alpha, \beta \in E^*$ with $s(\alpha) = s(\beta)$, let $Z(\alpha, \beta)$ be the set

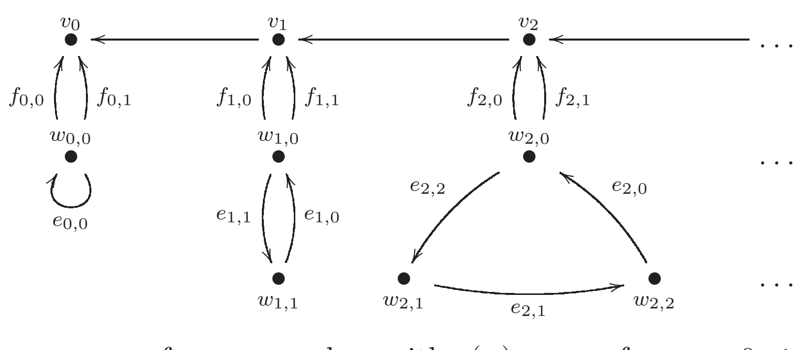
$$\{(x, k, y) : x \in Z(\alpha), y \in Z(\beta), k = |\beta| - |\alpha|, x_i = y_{i+k} \text{ for } i > |\alpha|\}.$$

By [18, Proposition 2.6], the collection of sets

$$\{Z(\alpha, \beta) : \alpha, \beta \in E^*, s(\alpha) = s(\beta)\}$$

is a basis of compact, open sets for a second-countable, locally compact, Hausdorff topology on $\mathcal{G}_E$ such that $\mathcal{G}_E$ is an r-discrete groupoid with a Haar system of counting measures. After identifying each $(x, 0, x) \in \mathcal{G}_E^{(0)}$ with $x \in E^\infty$, [18, Proposition 2.6] says that the topology on $\mathcal{G}_E^{(0)}$ is identical to the topology on E^∞. We caution that in our notation (which is now standard) the sources and ranges are swapped from the notation used in [18].

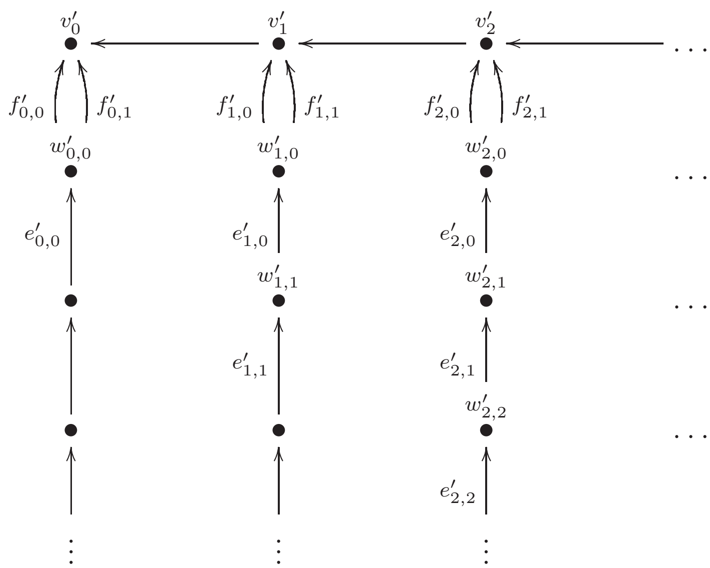
Example 7.1. Let E be the graph



Let $x \in E^\infty$. If $x = \alpha\alpha\alpha...$ for some cycle α with $r(\alpha) = w_{n,k}$ for some $0 \leq k \leq n$, then the isotropy subgroup of x in $\mathcal{G}_E$ is $A_x = (n+1)\mathbb{Z}$; otherwise $A_x = \{0\}$. It is straightforward to check that the isotropy subgroups vary continuously in the Fell topology. We claim

that the groupoid C^*-algebra $C^*(\mathcal{G}_E)$ has bounded trace but is not a Fell algebra. To see this, by Theorems 6.4 and 6.5, we need to show that $\mathcal{R} := \mathcal{G}_E/\mathcal{A}$ is integrable but not Cartan.

We start by considering the following graph F from [14, §8]:



There are no cycles in F, so $\mathcal{G}_F$ is a principal groupoid by [14, Proposition 8.1]. By [13], the groupoid $\mathcal{G}_F$ is integrable. So $C^*(\mathcal{G}_F)$ has bounded trace by [6, Theorem 4.4]. For $n \geq 0$, let x^n be the unique infinite path with range v_0' which has $f_{n,0}'$ as an edge, let y^n be the unique infinite path with range v_0' which has $f_{n,1}'$ as an edge, and let z be the infinite path going through each v_i'. It is shown in [14, Example 8.2] that the sequence $\{x^n\}$ converges 2-times in $\mathcal{G}_F^{(0)}/\mathcal{G}_F$ to z; the sequences in $\mathcal{G}_F$ witnessing this 2-times convergence are $\gamma_n^{(1)} = (x^n, 0, x^n)$ and $\gamma_n^{(2)} = (y^n, 0, x^n)$. It follows that $\mathcal{G}_F$ is not a Cartan groupoid by Lemma 5.1. Since $\mathcal{G}_F$ is principal, $C^*(\mathcal{G}_F)$ is not a Fell algebra by [4, Theorem 7.9].

Now consider the open subset

$$U = \bigcup_{i\geq 0} \big(Z(v_i') \cup_{j\leq i} Z(w_{i,j}') \big)$$

of F^∞. Let G be the groupoid obtained by restricting $\mathcal{G}_F$ to U. Then G is a principal, integrable groupoid which is not Cartan (because the two sequences witnessing the 2-times convergence of $\{x^n\}$ in $\mathcal{G}_F$ are also in G). Thus $C^*(G)$ has bounded trace but is not a Fell algebra.

We claim that G is isomorphic to $\mathcal{R} = \mathcal{G}_E/\mathcal{A}$. To see this, first note that "unwrapping" cycles in E^* sets up a bijection ϕ between E^* and the set of finite paths in U; similarly "unwrapping" cycles in E^∞ sets up a bijection ψ between E^∞ and the set of infinite paths in U. If α is a finite path in E^* then $Z(\phi(\alpha)) = \psi(Z(\alpha))$. Since the cylinder sets form a basis for the topology on E^∞, $\psi : E^\infty \to U$ is a homeomorphism.

Second, fix $(x, k, y) \in \mathcal{G}_E$ so that $x \sim_k y$. Then either (1) x and y are of the form $x = \alpha\gamma\gamma...$, $y = \beta\gamma\gamma....$ where γ is a cycle with $r(\gamma) = w_{n,0}$ for some $n \in \mathbb{N}$, and $\alpha, \beta \in E^*$ don't contain γ, or (2) both x and y do not contain cycles. In (1), $\psi(x)$ and $\psi(y)$ are

shift-equivalent with lag $|\alpha| - |\beta|$, and in (2) $\psi(x)$ and $\psi(y)$ are shift-equivalent with lag k. Thus, since G is principal, if $(x, k, y) \in \mathcal{G}_E$ then there exists a unique l_k such that $(\psi(x), l_k, \psi(y)) \in G$.

Finally, it is now straightforward to verify that

$$\rho : \mathcal{G}_E \to G \text{ defined by } \rho((x, k, y)) = (\psi(x), l_k, \psi(y))$$

is a groupoid homomorphism which is continuous, open and surjective, and that ρ induces a homeomorphism $\rho : \mathcal{R} \to G$. Thus $\mathcal{R}$ is an integrable groupoid which is not Cartan, and hence $C^*(\mathcal{G}_E)$ has bounded trace but is not a Fell algebra by Theorems 6.4 and 6.5.

Example 7.2. Let $\mathcal{G}_E$ be the groupoid from Example 7.1. Let $\mathcal{D}_E$ be the associated $\mathbb{T}$-groupoid over $\hat{\mathcal{A}} \rtimes \mathcal{R}$ defined by Muhly-Williams-Renault (see (6.3)). Note that the orbit space $\mathcal{G}_E^{(0)}/\mathcal{G}_E$ is T_1, so $C^*(\mathcal{G}_E)$ and $C^*(\mathcal{D}_E; \hat{\mathcal{A}} \rtimes \mathcal{R}, \delta \times \alpha)$ are isomorphic by Proposition 6.3. Thus $C^*(\mathcal{D}_E; \hat{\mathcal{A}} \rtimes \mathcal{R}, \delta \times \alpha)$ has bounded trace but is not a Fell algebra, and hence by Theorems 4.3 and 5.2, $\hat{\mathcal{A}} \rtimes \mathcal{R}$ is an integrable groupoid but is not Cartan.

Appendix A. Corrections to the proof of Theorem 2.3 of [19] contributed by Dana P. Williams.

Robert Hazlewood pointed out that there is a problem with the proof of Theorem 3.2 in [19]. On the bottom of page 237, we assert that we can find neighbourhoods V_0 and V_1 of z such that $\overline{V_0} \subseteq V_1$ with the property that[1]

$$\overline{W_1}^7 \overline{V_1}\, \overline{W_1}^7 \setminus W_0 V_0 W_0 \subseteq r^{-1}(G^{(0)} \setminus N). \tag{A.1}$$

Unfortunately, if $\overline{V_1}$ is larger than V_0, then we see no reason such neighbourhoods should exist. In fact, we now suspect that it is not possible to find such neighbourhoods — let alone via a "straightforward compactness argument". However, (A.1) does hold provided we restrict to elements with source sufficiently close to z.[2] Namely, we can prove the following.

Lemma A.1. *Given neighbourhoods V_0 and V_1 of z in G with V_0 open and V_1 relatively compact, there is a compact neighbourhood A of z in $G^{(0)}$ such that*

$$(\overline{W_1}^7 \overline{V_1} \setminus W_0 V_0) \cap G_A \subseteq r^{-1}(G^{(0)} \setminus N), \tag{A.2}$$

where $G_A := s^{-1}(A)$.

Proof. If no such A exists, we can let $\{ A_n \}$ be a neighbourhood basis of z with each A_n compact and $A_{n+1} \subseteq A_n$. Then, by assumption, for each n we can find $\gamma_n \in G_{A_n}$ belonging to the closed set $r^{-1}(N) \cap (\overline{W_1}^7 \overline{V_1} \setminus W_0 V_0)$. Since $\overline{W_1}^7 \overline{V_1}$ is compact, we can pass to a subsequence, relabel, and assume that $\gamma_n \to \gamma$. Notice that we must have $\gamma \in r^{-1}(N) \cap (\overline{W_1}^7 \overline{V_1} \setminus W_0 V_0)$. Since $s(\gamma_n) \in A_n$ and $s(\gamma_n) \to s(\gamma)$, we must have

[1] We are retaining the notations of [19] except we have dropped the fraktur font for groupoids and written G in place of $\mathfrak{G}$ for clarity. This is more of an issue in [20] where our readers have been frustrated trying to distinguish between $\mathfrak{G}$, $\mathfrak{S}$ and $\mathfrak{E}$ — rather than between G, S and E.

[2] A similar restriction was required in [29] — the function f_x^1 defined on the bottom of [29, p. 61] is only well-defined on U_0 (even though I failed to mention this). This is reflected in the statement of [29, Lemma 4.4].

$s(\gamma) = z$. Since $\gamma \notin W_0V_0$, we have $\gamma \in G_z \setminus W_0 z$. Since $F_z \subseteq W_0 z$, our construction of F_z forces $r(\gamma) \notin N$. But this is a contradiction. This completes the proof of the Lemma. □

Now, if $\gamma \in G_A$, then

$$g^{(1)}(\gamma) := \begin{cases} g\big(r(\gamma)\big) & \text{if } \gamma \in \overline{W_1}^7\,\overline{V_1}\,\overline{W_1}^7 \text{ and} \\ 0 & \text{if } \gamma \notin W_0V_0W_0 \end{cases} \tag{A.3}$$

is a well defined function on G_A. Consequently (A.3) defines an element of $C_c(G_A)$. We can use the Tietze-Extension Theorem to extend $g^{(1)}$ to an element of $C_c(G)$ provided we keep in mind that (A.3) holds only for $\gamma \in G_A$.

Next, we must modify [19, Lemma 2.8] to hold only near z; specifically, we have the following.

Lemma A.2. *With the choices above,*

$$g\big(r(\gamma)\big)g\big(r(\alpha)\big)b(\gamma\alpha^{-1})g^{(1)}(\alpha) = g^{(1)}(\gamma)g\big(r(\alpha)\big)g^{(1)}(\alpha)$$

provided $\gamma, \alpha \in G_A$.

Then with the given restriction on γ and α, the proof of Lemma A.2 goes through as written in [19]. Now it is straightforward to check that the rest of the proof of [19, Theorem 2.3] goes through with the observation that (1) we only need consider the representations L^u with u close to z, and that (2) L^u acts on $L^2(G_u, \lambda_u)$. This allows us to apply Lemma A.2 at the appropriate time.

Remark A.3. The existence of the neighbourhoods V_0 and V_1 such that (A.1) holds is used in the proof of [20, Theorem 2.3] (see page 237 of [20]) and in the proof of [6, Proposition 4.1]; both results are saved by Lemma A.1 since we can restrict to elements with source sufficiently close to z.

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